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## Properties of the Eleven-Dimensional Supermembrane Theory

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We study in detail the structure of the Lorentz covariant, spacetime supersymmetric 11-dimensional supermembrane theory. We show that for a flat spacetime background, the spacetime supersymmetry becomes an  $N=8$  world volume (rigid) supersymmetry in a “physical” gauge; we also present the field equations and transformation rules in a “lightcone” gauge. We semiclassically quantize the closed torodial supermembrane on a spacetime  $(Minkowski)_4 \times (flat\ 7-torus)$ , and review some mathematical results that are relevant for path integral quantization. © 1988 Academic Press, Inc.

### 1. INTRODUCTION

In their book on superstring theory [1] Green, Schwarz, and Witten remark that “eleven dimensional supergravity remains an enigma. It is hard to believe that its existence is just an accident, but it is difficult at the present time to state a compelling conjecture for what its role may be in the scheme of things.” In a previous paper [2] we suggested that, just as 10-dimensional supergravity is related to superstring theory, so 11-dimensional supergravity may be related to a supermembrane theory. In support of this connection we constructed an 11-dimensional supermembrane action, analogous to the Green–Schwarz (G–S) superstring action [3], and along the lines of the 6-dimensional 3-brane constructed by Hughes, Liu, and Polchinsky [4]. (We call a  $p$ -dimensional extended object a  $p$ -brane, so that a string is a 1-brane in this terminology.) We showed that the preservation of local symmetries of this action in an 11-dimensional background requires that the background satisfy certain constraints, which are equivalent to the equations of motion of 11-dimen-

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sional supergravity [2, 5]. Furthermore it has recently been argued that the spectrum of the 11-dimensional supermembrane contains the massless states of  $d=11$  supergravity [6]. These results suggest that study of supermembrane theories might provide an answer to the "enigma" of 11-dimensional supergravity. One can hope that a supermembrane theory will provide a quantum consistent extension of 11-dimensional supergravity just as superstring theories are thought to provide a quantum consistent extension of 10-dimensional supergravity theories.

The observation that led us in [2] to seek a connection of 11-dimensional supergravity with supermembranes was that the spectrum of 11-dimensional supergravity contains a third rank gauge potential which naturally couples to a membrane in the same way that the second rank antisymmetric tensor potential of 10-dimensional supergravity couples to a string. There are other supergravity theories with yet higher rank antisymmetric tensor potentials, which prompts the question of whether there are higher-dimensional supersymmetric extended objects, i.e., "super  $p$ -branes." There is also the question of which spacetime dimensions,  $d$ , are allowed for a given value of  $p$ . It is well known, for example, that the G-S superstring action exists for  $d=3, 4, 6, 10$  and one can show similarly that the supermembrane action of [2] exists for  $d=4, 5, 7, 11$ .

Recently a complete classification [7] of the values of  $p$  and  $d$  for which the super  $p$ -brane action exists has been given. As for the G-S superstring we require of the super  $p$ -brane action (which is an integral over the  $(p+1)$ -dimensional world volume) that it possess a certain fermionic gauge invariance allowing half of the fermionic variables to be gauged away. This invariance is crucial for many properties of the action. In particular, a most remarkable consequence is that (when all gauge invariances are taken into account) there are an equal number of  $((p+1)$ -dimensional) boson and fermion fields. Although these fermions are initially *spacetime* spinors they become *world volume* spinors after the world volume reparametrizations are fixed, as we shall show in this paper for the 11-dimensional supermembrane. Furthermore, for a supersymmetric spacetime background this leads to a (rigid) world volume supersymmetry ("supersymmetry on the brane"). Figure 1 summarizes the classification.

It has been shown in [5] that the G-S action for the type IIA superstring can be obtained from the 11-dimensional supermembrane by a process of double-dimensional reduction, whereby the dimension of spacetime,  $d$ , and the dimension of the extended object,  $p$ , are both reduced by one. One can see from the figure that each super  $p$ -brane action belongs to one of four sequences in which the members of a sequence are related in this way. The existence of these four sequences is related to the existence of the four composition-division algebras **R**, **C**, **H**, **O** [8, 7], so that we have called the four sequences containing the  $d=3, 4, 6, 10$  superstring the **R**-, **C**-, **H**-, **O**-sequences, respectively.

The super  $p$ -branes of [7] can be thought of *either* as "fundamental" extended objects in which the quantized excitations are considered as elementary particles, *or* as "cosmic"  $p$ -branes (i.e., extended object solutions of supersymmetric field theories) in the long wavelength limit [9]. One might see a "cosmic"  $p$ -brane

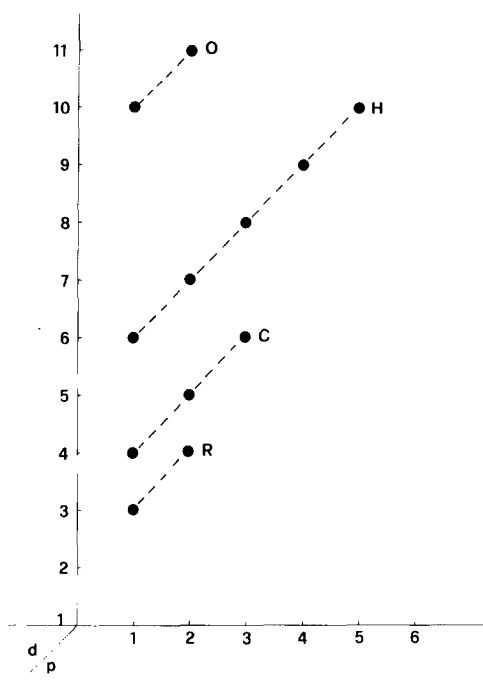


FIG. 1. Classification of allowed  $(p, d)$  values ( $1 \leq p \leq d-2$ ) for super  $p$ -branes. Only  $N=1$  spacetime supersymmetry is allowed for  $p \geq 2$ , while  $N=1, 2$  is allowed for  $p=1$ .

through a telescope but not a “fundamental”  $p$ -brane. Nevertheless, the same action can be used to describe both. For a fundamental  $p$ -brane there is, by hypothesis, no internal structure, while for a cosmic  $p$ -brane any information about the internal structure is lost in the long wavelength limit. In this paper we focus on the problems associated with the “fundamental super  $p$ -brane” interpretation for which the problem of quantum consistency is of course crucial.

Already classically the possibilities for super  $p$ -brane actions are severely limited. We might expect quantum considerations to impose yet further restrictions. Indeed we know that while the G-S superstring action exists for  $d=3, 4, 6, 10$  classically, it is only the  $d=10$  action that is quantum consistent (i.e., free from anomalies). This might lead one to suspect that the only quantum consistent super  $p$ -branes are those of the **O**-sequence, i.e., the  $d=10$  superstring and the  $d=11$  supermembrane. Indeed, it has recently been argued by Bars [10] that all of the super  $p$ -branes of the other three sequences have Lorentz anomalies (in the lightcone gauge), while for those of the **O**-sequence these anomalies may cancel (as, of course, they do for the superstring). Thus the only likely candidates for fundamental super  $p$ -branes would appear to be the  $d=10$  superstring and the  $d=11$  supermembrane. With these motivations in mind we present here many details of the properties of the 11-dimensional supermembrane.

In Section 2 we review the results of [2] but allow here for the possibility of an open supermembrane. The boundary conditions for an open supermembrane are complicated nonlinear equations. The fact that we are restricted to an  $N=1$  superspace means that these boundary conditions cannot be solved in the simple way of the open superstring. Partly for this reason most of this paper will be concerned with the closed supermembrane.

In Section 3 we present the equations of motion (in curved superspace) and we give the commutator algebra of the various symmetries. As for the G-S superstring, a commutator of two fermionic gauge transformations yields a world volume reparametrization plus another fermionic gauge transformation.

In Section 4 we present details of the action, etc., in flat 11-dimensional superspace. We discuss the symmetries of the equations of motion for this background and show that after the choice of a "physical" gauge the supermembrane equations are those of a 3-dimensional  $N=8$  supersymmetric field theory.

In Section 5 we show how to fix a "lightcone gauge." This is useful for comparison with the superstring and for the purposes of semiclassical quantization. We also give the supersymmetry, and other, transformations in this gauge.

In Section 6 we review the known classical solutions of the 11-dimensional supermembrane equations (the "brane-wave" equations) for various spacetime backgrounds. We also give a new solution, the supermembrane collapsed to a line, which generalizes the collapsed membrane solution of [6]. The considerations of Ref. [6] would also appear to apply to this solution, suggesting that a potential problem with the supermembrane is the occurrence of too many massless states rather than none at all.

The semiclassical quantization of the supermembrane about given classical solutions has been discussed in two recent papers [11, 12]. In Section 7 we generalize the results of [11] to the case of a toroidal membrane in an 11-dimensional background that is the product of 4-dimensional Minkowski spacetime and a 7-dimensional flat torus. As in [11] we find that the vacuum energy cancels, as a result of the supersymmetry of the background.

The full quantization of the supermembrane is, of course, a much harder problem. We discuss some of the issues in Section 8. One approach would be via a functional integral of a 3-dimensional sigma model over all possible 3-manifolds with suitable boundary conditions. With future applications of this approach in mind we collect some mathematical results on the geometry and topology of 3-manifolds.

## 2. SUPERMEMBRANE IN CURVED SUPERSPACE

In [2] we proposed the supermembrane action

$$I = \int d^3\xi \left[ -\frac{1}{2} \sqrt{-g} g^{ij} \Pi_i^a \Pi_j^b \eta_{ab} + \frac{1}{2} \sqrt{-g} - \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right], \quad (2.1)$$

where  $\xi^i$  ( $i=0, 1, 2$ ) are the world volume coordinates,  $g_{ij}$  is the metric of the world volume,  $g = \det(g_{ij})$ ,  $\eta_{ab}$  is the  $d=11$  Minkowski spacetime metric, and

$$\Pi_i^A = \partial_i Z^M E_M^A, \quad A = a, \alpha; \quad M = \mu, \dot{\alpha}. \quad (2.2)$$

Here,  $Z^M$  are the coordinates of the  $d=11$  curved superspace, and  $E_M^A$  is the supervielbein. The 3-form

$$B = \frac{1}{6} E^A E^B E^C B_{CBA}, \quad E^A = dZ^M E_M^A \quad (2.3)$$

is the potential for the closed 4-form

$$H = dB. \quad (2.4)$$

We refer the reader to the Appendix for details of the conventions.

The action  $I[Z^M, g_{ij}]$  has the local gauge invariances:

$d=3$  reparametrizations:

$$\delta Z^M = \eta^i(\xi) \partial_i Z^M, \quad \delta g_{ij} = \eta^k \partial_k g_{ij} + 2\partial_{(i} \eta^k g_{j)k}. \quad (2.5)$$

$d=3$  local fermionic “ $\kappa$ -transformations”:

$$\delta Z^M E_M^a = 0, \quad (2.6)$$

$$\delta Z^M E_M^\alpha = (1 + \Gamma)^\alpha{}_\beta \kappa^\beta, \quad (2.7)$$

$$\begin{aligned} \delta(\sqrt{-g} g^{ij}) = & -2i(1 + \Gamma)^\alpha{}_\beta \kappa^\beta (\Gamma_{ab})_{\alpha\gamma} \Pi_n^\gamma g^{n(i} \varepsilon^{j)kl} \Pi_k^a \Pi_l^b \\ & - \frac{2i}{3\sqrt{-g}} \kappa^\alpha (\Gamma_c)_{\alpha\beta} \Pi_\kappa^\beta \Pi_l^c g^{kl} \varepsilon^{mn(i} \varepsilon^{j)pq} \\ & \times (\Pi_m^a \Pi_{pa} \Pi_n^b \Pi_{qb} + \Pi_m^a \Pi_{pa} g_{nq} + g_{mp} g_{nq}), \end{aligned} \quad (2.8)$$

where  $\kappa^\alpha(\xi)$  is an *anticommuting* spacetime spinor, and the matrix  $\Gamma$  is defined by

$$\Gamma = \frac{1}{6\sqrt{-g}} \varepsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{abc}. \quad (2.9)$$

Here we are taking the  $d=3$  world volume metric  $g_{ij}$  to be an independent variable. It is often convenient to use the “Dirac form” of the action [13], which is obtained by using the equation of motion for  $g_{ij}$ ,

$$g_{ij} = \Pi_i^a \Pi_j^b \eta_{ab}, \quad (2.10)$$

and substituting for  $g_{ij}$  in the action (2.1). In this case we need not specify  $\delta g_{ij}$ ; its variation is determined by that of  $Z^M$  and differs from (2.8) by field equation terms. Using (2.8) one easily shows that

$$\Gamma^2 = 1, \quad (2.11)$$

so that the matrix  $(1 + \Gamma)$  occurring in (2.7) is a projection operator. As a consequence, the  $\kappa$ -transformation allows only *half* of the  $\theta$ 's to be gauged away.

If the action is considered not only as a functional of the 3-dimensional fields  $Z^M(\xi)$ ,  $g_{ij}(\xi)$ , but also as a function of the spacetime background  $E_M^A$ ,  $B_{MNP}$  (i.e., as a function of its "coupling constants" as well as a functional of fields), then it is also invariant under the *spacetime* gauge transformations:

$d = 11$  superspace transformations:

$$\begin{aligned}\delta Z^M &= -K^M(Z), & \delta E_M^A &= K^N \partial_N E_M^A + \partial_M K^N E_N^A, \\ \delta B_{MNP} &= K^Q \partial_Q B_{MNP} + \partial_M K^Q B_{QNP} + 2 \text{ more terms},\end{aligned}\quad (2.12)$$

$d = 11$  superspace tensor gauge transformations:

$$\delta B_{MNP} = \partial_M \Sigma_{NP}(Z) + 2 \text{ more terms}, \quad (2.13)$$

discrete transformation:

$$B_{MNP} \rightarrow -B_{MNP}, \quad (2.14)$$

together with an odd number of (spacetime) space or time reflections.

Of course, the superspace coordinate transformation is an invariance of  $I[Z, g_{ij}]$  of the conventional type (for which there is a Noether current) only when  $K^M(Z)$  and the spacetime background are such that  $K^M$  is a Killing supervector generating an isometry of the background and  $H_{ABCD}$  is an invariant tensor of the isometry supergroup.

The invariances (2.5)–(2.8) require the parameters  $\eta, \kappa$  to be such that

$$\int d^3\xi \partial_i (\eta^i \mathcal{L}) = 0, \quad (2.15)$$

$$\int d^3\xi \partial_i [\varepsilon^{ijk} (1 + \Gamma)^\alpha{}_\beta \kappa^\beta \Pi_j^A \Pi_k^B B_{BA\alpha}] = 0, \quad (2.16)$$

which are immediately satisfied by a closed supermembrane if  $\eta, \kappa$  vanish at the initial and final times. The spacetime  $\Sigma$ -invariance requires that

$$\int d^3\xi \partial_i (\varepsilon^{ijk} \Pi_j^A \Pi_k^B \Sigma_{BA}) = 0. \quad (2.17)$$

In addition, as we showed in [2] the  $\kappa$ -invariance of the supermembrane action (2.1) imposes the following constraints on the  $d = 11$  supergeometry:

$$T^a{}_{\alpha\beta} = -2i(\Gamma)^\alpha{}_{\alpha\beta}, \quad (2.18)$$

$$H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma d} = 0, \quad H_{\alpha\beta ab} = \frac{-i}{3} (\Gamma_{ab})_{\alpha\beta}, \quad (2.19)$$

$$\eta_{c(a} T_{b)\alpha}^c = \eta_{ab} A_\alpha, \quad H_{\alpha abc} = \frac{-1}{2} A_\beta (\Gamma_{abc})^\beta{}_\alpha. \quad (2.20)$$

Here,  $A_\alpha$  is an arbitrary spinor field. At first sight the conditions for  $\kappa$ -invariance appear weaker than those of the  $d=11$  on-shell torsion constraints of [14, 15], but it has been shown recently [5] that the “additional” freedom allowed by  $\kappa$ -invariance is just the freedom allowed in the choice of the conventional constraints of  $d=11$  supergravity. More precisely, by suitable redefinitions of the superconnection and parts of the supervielbein, we may set  $A_\alpha$ ,  $T_{\beta\gamma}^\alpha$ , and  $T_{bc}^a$  equal to zero:

$$T_{b\gamma}^a = T_{\beta\gamma}^\alpha = T_{bc}^a = H_{\alpha abc} = 0. \quad (2.21)$$

Equations (2.18)–(2.19), and (2.21) are the standard constraints of  $d=11$  on-shell super-gravity. Substituting these constraints into the Bianchi identities

$$DT^A = E^B R_B^A, \quad dH = 0, \quad (2.22)$$

one can show [14, 15] that all torsions, curvatures, and components of the 4-form  $H$  can be expressed in terms of a single superfield  $H_{abcd}$  which satisfies the constraints

$$D_{[a} H_{bcde]} = 0, \quad (2.23)$$

$$D_\alpha H_{abcd} = \frac{-i}{7} (\Gamma_{[ab} \Gamma^{ef})^\beta{}_\alpha D_\beta H_{cd]}{}^{ef},$$

where  $D_\alpha = E_\alpha^M \partial/\partial Z^M$  is the usual spinorial derivative. The independent components of  $H_{abcd}$  are  $H_{abcd}$  ( $\theta=0$  component), the gravitino curvature ( $\theta$  component), and the Riemann curvature tensor ( $\theta^2$  component). The remaining torsion components are given in terms of the basic superfields as follows:

$$T_{\beta a}^\gamma = \frac{1}{6} H_{abcd} (\Gamma^{bcd})^\gamma{}_\beta - \frac{1}{48} (\Gamma_{abcde})^\gamma{}_\beta H^{bcde}, \quad (2.24)$$

$$T_{ab}^\alpha = \frac{i}{14} (\Gamma^{cd})^{\alpha\beta} D_\beta H_{abcd}. \quad (2.25)$$

An important feature of the superspace geometry of  $d=11$  supergravity is the existence of the closed 4-form  $H$ .

$$H = \frac{i}{12} E^a E^b E^\alpha E^\beta (\Gamma_{ab})_{\alpha\beta} + \frac{1}{24} E^a E^b E^c E^d H_{abcd}. \quad (2.26)$$



In order to show that  $dH=0$ , one needs (2.18) and (2.21), and the identity

$$\eta_{ab} \Gamma_{\{\alpha\beta}^a \Gamma_{\gamma\delta\}}^{bc} = 0. \quad (2.27)$$

This identity is the analogue, for the supermembrane, of the identity  $\eta_{ab} \Gamma_{\{\alpha\beta}^a \Gamma_{\gamma\delta\}}^b = 0$  for superstring in  $d=10$ .

The expressions for the supervielbein, the superconnection and the super 3-form, and the parameter  $K^M$ , to lowest order in  $\theta$  have been given in [14]. Using these results one can easily determine the  $\psi_\mu$ -dependence of  $\Pi_i^A$  to lowest order. One finds

$$\begin{aligned} \Pi_i^\mu &= \partial_i X^\mu - i\bar{\theta} \Gamma^\mu (\partial_i \theta + \psi_i), \\ \Pi_i^\alpha &= \partial_i \theta^\alpha + \psi_i^\alpha, \end{aligned} \quad (2.28)$$

where  $\psi_i^\alpha = \partial_i X^\mu \psi_\mu^\alpha$  is the pullback of the gravitino to the world volume. Observe that  $\psi_i^\alpha$  appears always in the combination  $(\psi_i + \partial_i \theta)$ , as one expects from the fact that  $\theta^\alpha$  (or rather half of its components, because of the  $\kappa$ -invariance) are Goldstone fermions for spacetime supersymmetry.

### 3. EQUATIONS OF MOTION AND THE COMMUTATOR ALGEBRA

The equations of motion which follow from the action (2.1) are the “embedding equation”

$$g_{ij} - \Pi_i^a \Pi_j^b \eta_{ab} = 0, \quad (3.1)$$

and the super “brane-wave” equations

$$\begin{aligned} \partial_i (\sqrt{-g} g^{ij} \Pi_j^a) + \sqrt{-g} g^{ij} \Pi_j^b \Pi_i^c \Omega_{cb}{}^a \\ + i\bar{e}^{ijk} \Pi_{ib} (\Pi_j^a \Gamma_{\alpha\beta}^{ab} \Pi_k^\beta) \\ + e^{ijk} \Pi_i^b \Pi_j^c \Pi_k^d H_{bcd}^a = 0, \end{aligned} \quad (3.2)$$

$$[(1 - \Gamma) g^{ij} \Pi_i^\mu \Gamma_\mu]^\alpha{}_\beta \Pi_j^\beta = 0, \quad (3.3)$$

where  $\Omega_B{}^A$  is the connection 1-form in  $d=11$  curved superspace. In deriving these field equations we have used the identity

$$\delta \Pi_i^A = \partial_i (\delta Z^A) - \delta Z^B \Pi_i^C T_{CB}^A + \delta Z^B \Pi_i^C \Omega_{CB}{}^A, \quad (3.4)$$

where

$$\delta Z^A \equiv \delta Z^M E_M{}^A. \quad (3.5)$$

The requirement that the action be stationary when the equations of motions (3.1)–(3.3) hold imposes on  $Z^M(\xi)$  the boundary condition

$$\int d^3\xi \partial_i (-\delta Z^a \sqrt{-g} g^{ij} \Pi_{ja} - 3e^{ijk} \delta Z^A \Pi_j^B \Pi_k^C B_{CBA}) = 0. \quad (3.6)$$

As explained in the previous section the basic superfield  $H_{abcd}$ , whose independent components are the fields of the  $d=11$  supergravity (i.e., the graviton, the gravitino, and the three-index antisymmetric tensor field), is *not* arbitrary: already at the classical level the  $\kappa$ -invariance imposes constraints on  $H_{abcd}$ , which are equivalent to the field equations of  $d=11$  supergravity. Precisely, from (2.18), (2.19), and (2.21) it follows that

$$R_{ab} = 3 \left( H_{acde} H_b^{cde} - \frac{1}{12} \eta_{ab} H_{cdef} H^{cdef} \right) \quad (3.7)$$

$$\mathcal{D}^a H_{abcd} = -\frac{1}{192} \varepsilon_{bcde_1 \dots e_4 f_1 \dots f_4} H^{e_1 \dots e_4} H^{f_1 \dots f_4}, \quad (3.8)$$

$$(\Gamma^{abc})^\alpha{}_\beta T_{bc}^\beta = 0, \quad (3.9)$$

where the  $\theta=0$  component of  $T_{ab}^\alpha$  is the gravitino curvature. The component field equations of  $d=11$  supergravity are just the  $\theta=0$  parts of (3.7)–(3.9).

We now turn to the computation of the on-shell commutator algebra of the  $\kappa$ -transformations and the world volume reparametrizations. To this end it is convenient to introduce the notation

$$\Gamma^a = g^{ij} \Pi_i^a \tau_j \Gamma + \gamma^a, \quad (3.10)$$

where

$$\tau_i \equiv \Pi_i^a \Gamma_a \Gamma, \quad (3.11)$$

$$\gamma^a \equiv \Gamma^a - g^{ij} \Pi_i^a \Pi_j^b \Gamma_b. \quad (3.12)$$

Note that the  $\gamma^a$  ( $a=0, \dots, 10$ ) are not linearly independent, since, using (3.1), one finds

$$\Pi_i^a \gamma_a = 0. \quad (3.13)$$

Using (3.1), one can also establish the following relation:

$$[\Gamma, \tau_i] = \{\Gamma, \gamma^a\} = [\tau_i, \gamma^a] = 0, \quad (3.14)$$

$$\{\tau_i, \tau_j\} = 2g_{ij}, \quad (3.15)$$

$$\{\gamma^a, \gamma^b\} = 2(\eta^{ab} - g^{ij} \Pi_i^a \Pi_j^b), \quad (3.16)$$

$$\frac{1}{2\sqrt{-g}} \varepsilon^{ijk} \tau_{jk} = \tau^i, \quad \frac{1}{\sqrt{-g}} \varepsilon^{ijk} \tau_k = -\tau^{ij}, \quad (3.17)$$

where  $\tau_{ij} \equiv \Pi_i^a \Pi_j^b \Gamma_{ab}$ .

Using (3.10)–(3.17), we derive the on-shell commutator algebra

$$[\delta(\eta_1), \delta(\eta_2)] = \delta(\eta_3 = \eta_2^i \partial_j \eta_1^i - (1 \leftrightarrow 2)), \quad (3.18)$$

$$[\delta(\kappa), \delta(\eta)] = \delta(\kappa' = \eta^i \partial_i \kappa), \quad (3.19)$$

$$\begin{aligned} [\delta(\kappa_1), \delta(\kappa_2)] = & \delta(\kappa_3 = -2i\gamma^a \Pi_i(\bar{\kappa}_2^- \gamma_a \tau^i \kappa_1^+) + 4i\Pi_i(\bar{\kappa}_2^+ \tau^i \kappa_1^+)) \\ & + \delta(\eta^i = -8i\bar{\kappa}_2^+ \tau^i \kappa_1^+), \end{aligned} \quad (3.20)$$

where

$$\kappa^\pm = \frac{1}{2}(1 \pm \Gamma)\kappa. \quad (3.21)$$

In computing the commutator algebra we have made use of the relations

$$\delta_\kappa \Gamma = (\delta_\kappa \Pi_i^a) \gamma_a \tau^i, \quad (3.22)$$

$$\delta_\kappa \Pi_i^a = -4(\bar{\kappa}^+ \Gamma^a \Pi_i), \quad (3.23)$$

$$\begin{aligned} \gamma_a \tau_i \lambda_1^+ (\bar{\lambda}_2^+ \gamma^a \lambda_3^-) - \gamma^a \lambda_1^+ (\bar{\lambda}_2^+ \gamma_a \tau_i \lambda_3^-) - (1 \leftrightarrow 2) \\ = 2\tau_{ij} \lambda_3^- (\bar{\lambda}_1^+ \tau^j \lambda_2^+). \end{aligned} \quad (3.24)$$

The relation (3.24) follows from (2.27). These results are qualitatively similar to those for the G-S superstring.

#### 4. THE SUPERMEMBRANE IN $d=11$ FLAT SUPERSPACE

Flat  $d=11$  superspace is a solution of the 11-dimensional supergravity superfield equations and is therefore a consistent background for the supermembrane. Let us denote the coordinates of the  $d=11$  flat superspace by  $Z^M = (X^\mu, \theta^\alpha)$ , where  $\theta^\alpha$  is a 32-component Majorana spinor. We need not distinguish here between  $\theta^\alpha$  and  $\theta^{\dot{\alpha}}$  because  $E_\alpha^z = \delta_\alpha^z$ . In fact in flat superspace  $E_M^A$  is given by

$$E_M^a = (\delta_\mu^a, -i(\Gamma^a)_{\alpha\beta} \theta^\beta), \quad E_M^\alpha = (0, \delta_\beta^\alpha). \quad (4.1)$$

In addition  $H_{abcd} = 0$ , and consequently  $dB = H$  with all components of  $H$  vanishing except  $H_{ab\alpha\beta} = -\frac{1}{3}(\Gamma_{ab})_{\alpha\beta}$ . Solving for  $B$ , one finds

$$B_{\mu\nu\rho} = 0, \quad B_{\mu\nu\alpha} = -\frac{i}{6}(\Gamma_{\mu\nu}\theta)_\alpha, \quad (4.2)$$

$$B_{\mu\alpha\beta} = -\frac{1}{6} (\Gamma_{\mu\nu}\theta)_{(\alpha} (\Gamma^\nu\theta)_{\beta)}, \quad (4.3)$$

$$B_{\alpha\beta\gamma} = -\frac{i}{6} (\Gamma_{\mu\nu}\theta)_{(\alpha} (\Gamma^\mu\theta)_{\beta} (\Gamma^\nu\theta)_{\gamma)}. \quad (4.4)$$

From (4.1)–(4.4) it follows that

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\theta}\Gamma^\mu \partial_i \theta, \quad (4.5)$$

$$\Pi_i^z = \partial_i \theta^z. \quad (4.6)$$

Using (4.1)–(4.4) in the supermembrane action (2.1) we obtain

$$\begin{aligned} I_{\text{flat}} = & -\frac{1}{2} \int d\xi \left[ \sqrt{-g} g^{ij} \Pi_i^\mu \Pi_{j\mu} - \sqrt{-g} \right. \\ & \left. + i\varepsilon^{ijk} \bar{\theta} \Gamma_{\mu\nu} \partial_i \theta \left( \Pi_j^\mu \Pi_k^\nu + i\Pi_j^\mu \bar{\theta} \Gamma^\nu \partial_k \theta - \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_j \theta \bar{\theta} \Gamma^\nu \partial_k \theta \right) \right]. \end{aligned} \quad (4.7)$$

Because flat superspace has the super Poincaré group as its isometry supergroup this action possesses (rigid) spacetime translation, Lorentz, and supersymmetry invariances in addition to the (local) reparametrizations and  $\kappa$ -invariances. The combined *fermionic* transformations are

$$\delta X^\mu = i\bar{\theta}\Gamma^\mu(1 + \Gamma)\kappa - i\bar{\theta}\Gamma^\mu\varepsilon, \quad (4.8)$$

$$\delta\theta = (1 + \Gamma)\kappa + \varepsilon, \quad (4.9)$$

$$\begin{aligned} \delta(\sqrt{-g} g^{ij}) = & 2i\bar{\kappa}(1 + \Gamma) \Gamma_{\mu\nu} \partial_n \theta g^{n(i} \varepsilon^{j)kl} \Pi_k^\mu \Pi_l^\nu \\ & + \frac{2i}{3\sqrt{-g}} \bar{\kappa} \Gamma^\mu \partial_l \theta \Pi_\mu^l \varepsilon^{mn(i} \varepsilon^{j)pq} (\Pi_m^\nu \Pi_{p\nu} \Pi_n^\lambda \Pi_{q\lambda} \\ & + \Pi_m^\nu \Pi_{p\nu} g_{nq} + g_{mp} g_{nq}), \end{aligned} \quad (4.10)$$

where  $\kappa = \kappa(\xi)$  is the parameter of the *local* fermionic transformation, and  $\varepsilon$  is the constant parameter of *rigid* supersymmetry transformations; both are 32-component Majorana spinors and world volume scalars. The combined *bosonic* transformations are given by

$$\delta X^\mu = \eta^i \partial_i X^\mu + l_{\nu}^\mu X^\nu + a^\mu, \quad (4.11)$$

$$\delta\theta = \eta^i \partial_i \theta + \frac{1}{4} l_{\mu\nu} \Gamma^{\mu\nu} \theta, \quad (4.12)$$

$$\delta(\sqrt{-g} g^{ij}) = \partial_k (\sqrt{-g} g^{ij} \eta^k) - 2\sqrt{-g} g^{k(i} \partial_k \eta^{j)}, \quad (4.13)$$

where  $\eta = \eta(\xi)$  is the parameter of the general coordinate transformations (i.e., reparametrizations) of the world volume, and  $(l_{\mu\nu} = -l_{\nu\mu}, a^\mu)$  are the constant

parameters of the  $d = 11$  rigid Poincaré transformations. The  $\kappa$ -transformations and the world volume reparametrizations close on-shell as in (3.18)–(3.20), with  $\Pi_i^A$  given in (4.5)–(4.6).

The invariances (4.8)–(4.13) require

$$\int d^3\xi \partial_i (\eta^i \mathcal{L}) = 0, \quad (4.14)$$

$$\int d^3\xi \partial_i [(1 + \Gamma)\kappa]^x S_\alpha^i = 0, \quad (4.15)$$

which restrict the parameters  $\eta$  and  $\kappa$  at the boundary, and

$$\begin{aligned} \int d^3\xi \partial_i \varepsilon^{ijk} \left[ \frac{-i}{2} \bar{\varepsilon} \Gamma_{\mu\nu} \theta \left( \Pi_j^\mu \Pi_k^\nu + \frac{5i}{3} \Pi_j^\mu \bar{\theta} \Gamma^\nu \partial_k \theta - \frac{11}{15} \bar{\theta} \Gamma^\mu \partial_j \theta \bar{\theta} \Gamma^\nu \partial_k \theta \right) \right. \\ \left. + \frac{1}{6} \bar{\varepsilon} \gamma^\nu \theta \bar{\theta} \Gamma_{\mu\nu} \partial_j \theta \left( \Pi_k^\mu + \frac{4i}{5} \bar{\theta} \gamma^\mu \partial_k \theta \right) \right] = 0, \end{aligned} \quad (4.16)$$

in order that the action be supersymmetric. The quantity  $S_\alpha^i$  appearing in (4.15) is

$$\begin{aligned} S_\alpha^i &\equiv 3\varepsilon^{ijk} \Pi_j^A \Pi_k^B B_{BA\alpha} \\ &= \varepsilon^{ijk} \left[ \frac{i}{2} (\Gamma_{\mu\nu} \theta)_\alpha \left( \Pi_j^\mu \Pi_k^\nu + i \Pi_j^\mu \bar{\theta} \Gamma^\nu \partial_k \theta + \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_j \theta \bar{\theta} \Gamma^\nu \partial_k \theta \right) \right. \\ &\quad \left. - \frac{1}{2} (\gamma^\nu \theta)_\alpha \bar{\theta} \Gamma_{\mu\nu} \partial_j \theta \left( \Pi_k^\mu - \frac{2i}{3} \bar{\theta} \gamma^\mu \partial_k \theta \right) \right]. \end{aligned} \quad (4.17)$$

The equations of motion which follow from (4.7) are

$$g_{ij} = \Pi_i^\mu \Pi_{j\mu}, \quad (4.18)$$

$$\partial_i (\sqrt{-g} g^{ij} \Pi_j^\mu) + \varepsilon^{ijk} \Pi_i^\nu \partial_j \bar{\theta} \Gamma_{\nu}^\mu \partial_k \theta = 0, \quad (4.19)$$

$$(1 - \Gamma) g^{ij} \Pi_i^\mu \Gamma_{\mu}^\nu \partial_j \theta = 0, \quad (4.20)$$

and the boundary condition (3.6) now reads

$$\int d^3\xi \partial_i (-\delta Z^M E_M^\mu P_\mu^i - \delta Z^M E_M^\alpha S_\alpha^i) = 0, \quad (4.21)$$

where

$$P^{\mu i} = \sqrt{-g} g^{ij} \Pi_j^\mu - i \varepsilon^{ijk} (\bar{\theta} \Gamma^{\mu\nu} \partial_j \theta) \left( \Pi_{k\nu} + \frac{i}{2} \bar{\theta} \Gamma_\nu \partial_k \theta \right). \quad (4.22)$$

Recall that for the open superstring we start with the type IIB superspace which has two Majorana–Weyl spinorial coordinates  $\theta^1$  and  $\theta^2$  of the same chirality. The

closed 3-form used in this case for the Wess–Zumino term in the action is  $E^a(d\bar{\theta}^1 \Gamma_a d\theta^1 - d\bar{\theta}^2 \Gamma_a d\theta^2)$  and the fermionic boundary term can be made to cancel by setting  $\theta^1 = \theta^2$  on the boundary. This also has the effect of breaking  $N=2$  to  $N=1$  supersymmetry [3]. In our case no such simple means of satisfying the fermionic boundary condition exists because only  $N=1$  superspace allows a  $\kappa$ -invariant action in  $d=11$  [7]. The only way we know of satisfying the fermionic boundary condition is to set  $\theta=0$  at the boundary, but this would break supersymmetry. For this reason we shall not consider further the open supermembrane. Given that the only massless supersymmetric multiplet in  $d=11$  is that of  $d=11$  supergravity it would not be surprising if only the closed supermembrane were consistent. This is to be contrasted with  $d=10$  where both a super Yang–Mills and a supergravity multiplet exist.

The super brane-wave equations can be expressed as

$$B^\mu = \partial_i P^{i\mu} = 0, \quad (4.23)$$

$$F = \sqrt{-g} (1 - \Gamma) \tau^i \partial_i \theta = 0. \quad (4.24)$$

Using (4.18), one can show that there are only eight independent bosonic fields because of the identity

$$\Pi^\mu B_\mu = -2i(\partial_i \bar{\theta}) F. \quad (4.25)$$

Similarly, there are eight independent fermi fields (on-shell) since  $(1 - \Gamma)$  and  $\tau^i \partial_i$  act as projection operators. The quantities  $B^\mu$  and  $F$  are invariant under the rigid  $\varepsilon$ -supersymmetry transformations, while under the local  $\kappa$ -transformations they transform into each other as

$$\delta B^\mu = -2i \partial_i (\bar{\kappa}^+ \gamma^\mu \tau^i F), \quad (4.26)$$

$$\delta F = -2B^\mu \gamma_\mu \kappa^+ - 2i\gamma_\mu (\partial_i \theta)^+ (\bar{\kappa}^+ \gamma^\mu \tau^i F) - 2i\gamma_\mu \tau^i F (\bar{\kappa}^+ \gamma^\mu \partial_i \theta), \quad (4.27)$$

where  $\kappa^\pm$  are defined in (3.21). Observe that the flat superspace action (4.7) can be expressed as

$$I_{\text{flat}} = \int d^3 \xi [X_\mu B^\mu + i\bar{\theta} F - \partial_i (X_\mu P^{i\mu})]. \quad (4.28)$$

The action and transformation rules can be written explicitly in terms of the  $8+8$  physical bosons and fermions by a choice of “physical” gauge. One such choice is to identify  $\xi^i$  with  $x^0$  and two spatial coordinates. Let us make the  $3+8$  split

$$X^\mu \rightarrow (X^i, X^I) \quad (i=0, 9, 10; I=1, 2, \dots, 8) \quad (4.29)$$

(note the deviation from the conventions summarized in the Appendix), and choose the gauge

$$X^i = \xi^i. \quad (4.30)$$

Such a gauge is possible only locally for a closed membrane, although it would be possible globally for a membrane of infinite extent. This gauge has the advantage that it shows how the fermionic variables  $\theta$  which are initially world volume *scalars* become world volume *fermions* after gauge fixing.

From (4.11) we see that  $\eta^i$  must be fixed by

$$\eta^i(\xi) = -l^i_j \xi^j - l^i_j X^j(\xi) - a^i + \text{fermionic parameter terms}, \quad (4.31)$$

in order to maintain the gauge (4.30). Therefore, in this gauge,

$$\delta X^I = -l^i_j \xi^j \partial_i X^I + l^I_j X^J + \text{other parameter terms}. \quad (4.32)$$

The  $l^i_j$  transformations of the coset  $SO(10, 1)/[SO(2, 1) \times SO(8)]$  are now non-linear. The  $SO(8)$  transformation is linear and tells us that  $X^I$  is a vector of  $SO(8)$ . The  $SO(2, 1)$  transformation is just that of a world volume Lorentz scalar. Thus, after gauge fixing, the  $SO(2, 1)$  subgroup of the *spacetime Lorentz* group becomes the *world volume Lorentz* group [7]. The  $SO(2, 1) \times SO(8)$  transformation law for  $\theta$  in the gauge  $X^I = \xi^I$  is similarly found to be

$$\delta \theta = -l^i_j \xi^j \partial_i \theta + \frac{1}{4} l_{ij} \Gamma^{ij} \theta + \frac{1}{4} l_{IJ} \Gamma^{IJ} \theta, \quad (4.33)$$

which is just that of a *world volume* and  $SO(8)$  spinor. The  $\kappa$ -gauge for  $\theta$  can be fixed by the gauge choice

$$(1 + \Gamma_*)\theta = 0, \quad \Gamma_* = \Gamma^1 \Gamma^2 \dots \Gamma^8, \quad (4.34)$$

so that  $\theta$  becomes an  $SO(8)$  spinor of a definite chirality. Thus the physical fermion carries the  $(2, 8^s)$  representation of  $SO(2, 1) \times SO(8)$ . On fixing the  $\kappa$ -invariance the spacetime supersymmetry transformation must be accompanied by a compensating  $\kappa$ -transformation. Because of (4.26), (4.27) the combined rigid symmetry is one that transforms the boson and fermion field equations into one another. As the equations  $B^\mu = F = 0$  are nonlinear 3-dimensional field equations this symmetry is necessarily ( $N = 8$ ) supersymmetry. The bosonic action in the gauge (4.30) is simply

$$I = \int d^3 \xi \sqrt{\det(\eta_{ij} + \partial_i X^I \partial_j X^I)} \quad (4.35)$$

(on using the embedding equation as a constraint), and the supermembrane action can be thought of as an  $N = 8$  supersymmetric generalization of this.

In the next section we give the explicit form of the supersymmetry transformation rules in the “lightcone” gauge. Unlike the gauge  $X^I = \xi^I$  it does not allow the explicit

reduction to physical degrees of freedom only, and also breaks the  $SO(2, 1)$  invariance. The interpretation of these transformation rules as those of world volume supersymmetry is therefore quite obscure. The lightcone gauge has other advantages, however.

## 5. THE LIGHTCONE GAUGE

For the superstring it is very useful to impose the non-covariant "lightcone gauge conditions." We propose the following analogue of this lightcone gauge for the supermembrane, generalizing the gauge conditions of the bosonic membrane [16, 11, 17],

$$X^+ = p^+ \tau, \quad (5.1)$$

$$g_{00} = -h, \quad (5.2)$$

$$g_{0a} = 0 \quad (a = 1, 2), \quad (5.3)$$

$$\Gamma^+ \theta = 0, \quad (5.4)$$

where  $h = -\det(g_{ab})$ . (For the lightcone conventions see the Appendix.) Substituting these gauge conditions into the embedding equation (4.18), we can solve for  $X^-$  as follows:

$$V_a \equiv \Pi_a^- - \frac{1}{p^+} \dot{X}^I \partial_a X_I = 0, \quad (5.5)$$

$$\Pi_0^- = \frac{-1}{2p^+} (\dot{X}^I \dot{X}_I + h), \quad I = 1 \dots 9. \quad (5.6)$$

Taking the curl of (5.5) leads to an important constraint [11] which is independent of  $X^-$  and is given by

$$\varepsilon^{ab} \partial_a \dot{X}^I \partial_b X_I + i \sqrt{2} p^+ \varepsilon^{ab} (\partial_a \bar{S} \partial_b S) = 0, \quad (5.7)$$

where  $S$  is a real 16-component  $SO(9)$  spinor defined by  $\theta = (0, S)$ , which is the solution of the gauge condition (5.4).

The 2-vector  $V_a$  can be decomposed into its divergence-free, curl-free, and harmonic parts. On a 2-manifold harmonic vectors are curl- and divergence-free vectors and there are  $2g$  of them on a membrane which is a compact Riemann surface of genus  $g$ . They correspond to  $2g$  noncontractible loops on the membrane. Thus, for a membrane of genus  $g$ , we have the additional global constraint

$$\oint V_a dl^{a(w)} = 0, \quad w = 1, 2, \dots, 2g. \quad (5.8)$$



We now consider the field equations. Substituting the gauge conditions (5.1)–(5.4) into the field equations (4.18)–(4.20) we obtain the result

$$h_{ab} = \partial_a X^I \partial_b X_I, \quad (5.9)$$

$$-\dot{X}^I + \partial_a (h h^{ab} \partial_b X^I) - \sqrt{2} i p^+ \varepsilon^{ab} (\partial_a \bar{S} \gamma^I \partial_b S) = 0, \quad (5.10)$$

$$\dot{S} + \varepsilon^{ab} \partial_a X^I \gamma_I \partial_b S = 0, \quad (5.11)$$

where  $h_{ab} \equiv g_{ab}$ . In obtaining these results, it is useful to realize that  $\bar{\theta} \Gamma^\mu \partial \theta = 0$ , unless  $\mu = -$  and  $\bar{\theta} \Gamma^{\mu\nu} \partial \theta = 0$ , unless  $\mu\nu = -I$ . Because of these facts all the higher order fermion terms are absent in (5.9)–(5.11).

It is important to consider the question of whether there are gauge transformations which leave the gauge conditions (5.1)–(5.4) invariant. Such transformations, if any, would be the residual symmetries of the equations of motion and the constraints. We now investigate this systematically. It is convenient to consider first the fermionic gauge condition (5.4). Let us decompose the fermionic parameters  $\kappa$  and  $\varepsilon$  as  $\kappa = (i\kappa_1, \kappa_2)$ , and  $\varepsilon = (i\alpha, \beta)$ . Demanding that the total variation of  $\Gamma^+ \theta = 0$  vanish we find the following condition on the  $\kappa$ -parameter, (In what follows we shall drop the Lorentz parameters  $l^{\pm I}$  and  $l^{+-}$  because they play no further role.)

$$\kappa_2 = \frac{\sqrt{2}}{4p^+} [P\alpha + (P + 2\dot{X}^I \gamma_I) \kappa_1], \quad (5.12)$$

where

$$P = \varepsilon^{ab} \partial_a X^I \partial_b X^J \gamma_{IJ}. \quad (5.13)$$

Note that  $\kappa_1$  remains arbitrary. The  $\kappa_1$ -transformation is physically irrelevant, however. To see this, we substitute the  $\kappa_2$ -parameter as given in (5.12) into the transformation rules (4.8)–(4.10), and observe that all the  $\kappa_1$ -dependent terms drop out.

We next require that the total variation of the gauge condition (5.1) vanish. This fixes the time reparametrization as

$$\eta^0 = -\frac{a^+}{p^+}. \quad (5.14)$$

The requirement that the total variation of the gauge condition (5.3) vanish fixes the time dependence of the spatial reparametrizations in the following way (see (4.8) and (4.11)):

$$\dot{\eta}^a = 2ie^{ab} (\bar{\alpha} \partial_b S). \quad (5.15)$$

We finally consider the Hoppe gauge (5.2). Its total variation leads to the condition

$$\partial_b \eta^b = 0. \quad (5.16)$$

In obtaining (5.15) and (5.16), we have used the  $\kappa_2$ -parameter as given in (5.12), and the  $S$ -field equation (5.11).

We now observe that the conditions (5.15) and (5.16) do not fix completely the parameter  $\eta^a$ . From (5.16) it follows that reparametrizations of the form

$$\eta^a = \varepsilon^{ab} \partial_b f(\sigma, \rho) + \sum_{w=1}^{2g} c_w \eta_H^{a(w)} \quad (5.17)$$

for arbitrary  $f(\sigma, \rho)$ , are not fixed. Here,  $c_w$  are arbitrary constants, and  $\eta_H^{a(w)}$  is the harmonic part of  $\eta^a$ . The analogue of the transformation (5.17) for a closed string is the constant  $\sigma$ -reparametrization invariance. It is convenient *not* to fix the time independent reparametrizations, (5.17), of the membrane, until we linearize the theory in a semiclassical approximation scheme.

We are now in a position to give the final lightcone rigid transformation rules. Using (5.12) and (5.15) in the fermionic transformation rules (4.8)–(4.10), we find

$$\delta X^I = 2i\bar{\alpha}\gamma^I S + 2i\varepsilon^{ab} \partial_a X^I \bar{\alpha} \int_0^\tau d\tau \partial_b S, \quad (5.18)$$

$$\delta S = -\frac{\sqrt{2}}{2p^+} \left( \dot{X}^I \gamma_I - \frac{1}{2} \varepsilon^{ab} \partial_a X^I \partial_b X^J \gamma_{IJ} \right) \alpha + 2i\varepsilon^{ab} \partial_a S \bar{\alpha} \int_0^\tau d\tau \partial_b S + \beta. \quad (5.19)$$

Note that from the  $d=3$  point of view, there are two rigid supersymmetries, namely,  $\alpha$ - and  $\beta$ -transformations, and that the bosonic variable  $X^I$  is inert under the  $\beta$ -transformations. The latter are the nonlinearly realized part of an  $N=16$  rigid supersymmetry in the world volume. It is straightforward to derive the lightcone form of the bosonic transformations; the result is

$$\delta X^I = A^{IJ} X_J + a^I + \eta^i \partial_i X^I, \quad (5.20)$$

$$\delta S = \frac{1}{4} \ell^{IJ} \gamma_{IJ} S + \eta^i \partial_i S. \quad (5.21)$$

In (5.20) and (5.21), one must substitute for  $\eta^0$  the expression (5.14) and for  $\eta^a$  the solution of (5.15) and (5.16). The equations of motion and the constraints are also invariant under the time independent local reparametrization of the form (5.17).

Concerning the commutator algebra of the transformations (5.18)–(5.21), we remark that: (i) The  $\alpha$ - and  $\beta$ -transformations commute. (ii) The commutator of two  $\beta$ -transformations gives a constant translation in  $d=11$ . (iii) The commutator of two  $\alpha$ -transformations gives a constant time reparametrization in the world volume, and a local membrane reparametrization of the form (5.17). These membrane reparametrizations close among themselves.

## 6. CLASSICAL SOLUTIONS

The equations of motion of the supermembrane are intrinsically nonlinear (unlike those of the superstring) and we cannot hope to find the general solution. We can look for specific classical solutions, however, and then consider semiclassical quantization about them. The first step is to choose a background spacetime that solves the  $d=11$  supergravity field equations. The next step is to solve the bosonic and fermionic branewave equations (we consider the embedding equation as a constraint that eliminates  $g_{ij}$  as an independent variable). The fermionic equation is always solved by  $\theta = \text{const.}$  and the bosonic equation then reduces to that of the bosonic membrane. Given a solution of the latter we can determine whether it is supersymmetric by solving  $\delta S = 0$  (with  $\delta S$  given by (5.19)) for  $\alpha, \beta$ . If any non-zero  $\alpha, \beta$  exist for which  $\delta S = 0$  we say that the solution is supersymmetric. In this case one expects the Casimir energy of the quantum fluctuations about such solutions to vanish. This expectation is borne out by studies of semiclassical quantization to date [11, 12], and also by the results of Section 7. In this section we review the known classical solutions for various spacetime backgrounds and present a new one, the membrane collapsed to a line.

(1) *String Solutions.* If the  $d=11$  spacetime is  $(Mink)_{10} \times T^1$  then, as shown in [5], the double dimensional reduction ansatz

$$X^0 = \rho, \quad \partial_\rho X^{\hat{\mu}} = 0, \quad \hat{\mu} = 0, 1, \dots, 9, \quad (6.1)$$

for a toroidal membrane, reduces the membrane equations to those of the string. Therefore, any solution of the string equations, together with  $X^{10} = \rho$ , solves the membrane equations.

(2) *Static Toroidal Membrane.* By a further double-dimensional reduction ansatz

$$X^{10} = \rho, \quad X^9 = \sigma, \quad \partial_\rho X^{\hat{\mu}} = 0, \quad \hat{\mu} = 0, 1, \dots, 8, \quad (6.2)$$

the membrane equations reduce to those of a massive point particle. This mass  $m$  is just the energy stored in the surface of the toroidal membrane (i.e., the surface tension times the area). A solution to the membrane equation is therefore

$$X^{\hat{\mu}}(\tau) = a^{\hat{\mu}} + p^{\hat{\mu}}\tau, \quad p^2 = m^2, \quad (6.3)$$

together with (6.2) [11]. In the next section we discuss a generalization of this solution for the background  $(Mink)_4 \times T^7$ , and the semiclassical quantization about it. All solutions of this class are supersymmetric.

(3) *Spherical Solutions.* A spherical solution of the membrane equations was

considered in Dirac's original paper [13]. He found a static solution by supposing the membrane to carry an electric charge. In this case there is a critical radius for which the electrostatic repulsion balances the surface tension. For a membrane without an electric charge (which is the case of interest here) a static spherical solution is not possible. Collins and Tucker [18] considered a periodic pulsating membrane in which the spherical collapse to a point is followed by re-expansion. The solution is

$$X^0 = \tau, \quad X^1 = r(\tau) \sin \sigma \cos \rho, \quad X^2 = r(\tau) \sin \sigma \sin \rho, \quad X^3 = \dots = X^{10} = 0, \quad (6.4)$$

where  $r(\tau)$  satisfies

$$\dot{r} + \frac{1}{r_0^2} \sqrt{(r_0^4 - r^4)} = 0, \quad (6.5)$$

for any  $r_0 > 0$ . This equation can be solved in terms of special functions [18].

(4) *Pancake Membranes*. This was initially given as a solution of the open membrane by Kikkawa and Yamasaki [19] for a  $d$ -dimensional Minkowski spacetime with  $d \geq 5$ . The solution can be written as

$$X^0 = \tau, \quad X^1 + iX^2 = \sigma e^{i\omega\tau}, \quad X^3 + iX^4 = \rho e^{i\omega\tau}, \quad X^5 = \dots = X^{10} = 0. \quad (6.6)$$

It represents a disc spinning in the  $X^1 - X^2$  and  $X^3 - X^4$  planes. The boundary of an open membrane moves at the speed of light [18, 20] so the boundary of the pancake membrane is at  $(\rho^2 + \sigma^2) = \omega^{-2}$ .

This solution, which is clearly applicable to an 11-dimensional spacetime of the form  $(Mink)_d \times T_{(11-d)}$  for  $d \geq 5$ , was generalized to the closed membrane by Mezincescu *et al.* [12]; one takes two copies of the open membrane disc, one on top of the other, and identifies them along their boundaries. Observe that the (intrinsic) curvature of this closed membrane is singular at the boundary but as there is no curvature term in the action this is a regular solution of the closed membrane equations.

These ‘‘pancake’’ solutions are not supersymmetric, so that one does not expect the Casimir energy of the supermembrane fluctuations around this solution to vanish. Indeed, it does not vanish [12].

(5) *Hoppe–Nicolai Solutions*. These are closed surfaces of arbitrary genus  $g$  pulsating and rotating in  $d$ -dimensional Minkowski space [21] (for the supermembrane we take  $d = 11$ ). The solutions are given by

$$X^\mu = (\tau; r(\tau) \cos \phi(\tau) \mathbf{n}(\sigma, \rho), r(\tau) \sin \phi(\tau) \mathbf{n}(\sigma, \rho), 0, \dots, 0), \quad (6.7)$$

where  $r(\tau)$ ,  $\phi(\tau)$  are functions of  $\tau$  satisfying

$$r^2(\tau) \dot{\phi}(\tau) = \text{const} \neq 0, \quad (6.8)$$

and  $\mathbf{n}$  is a  $k$ -vector ( $3 \leq k \leq 5$ ), satisfying

$$\mathbf{n} \cdot \mathbf{n} = 1, \quad (\nabla^2 + 2) \mathbf{n} = 0, \quad (6.9)$$

with

$$\begin{aligned} \nabla^2 &= \frac{1}{\sqrt{\bar{h}}} \partial_a (\bar{h} \bar{h}^{ab} \partial_b), \\ \bar{h} &\equiv \det \bar{h}_{ab}, \quad \bar{h}_{ab} \equiv \partial_a \mathbf{n} \cdot \partial_b \mathbf{n}, \quad \partial_a \equiv (\partial_\sigma, \partial_\rho) \end{aligned} \quad (6.10)$$

(observe the deviation from the notation summarized in the Appendix). Equations (6.9) are the equations for minimal surfaces in  $S^{k-1}$ . For  $k=4$  such surfaces exist for any genus  $g$ . For  $g=0$  one can take [21]  $\mathbf{n} = (\sin \sigma \cos \rho, \sin \sigma \sin \rho, \cos \sigma, 0)$  for example. This gives a membrane that is homeomorphic to a 2-sphere, but which is not metrically a 2-sphere (and so does not come under the spherical solution rubric). Such membranes need not collapse to a point, just as non-planar-closed string loops need not collapse. We refer to [21] for further discussion of these solutions.

(6) *Collapsed Membranes.* In these solutions the membrane is collapsed to a configuration of zero area, i.e., a line or a point. The membrane collapsed to a point was considered in [6]. It is a special case of the membrane collapsed to a line, which is the new solution given here. Following [6] we observe that by a gauge choice we can arrange to put  $g_{ij}$  in the form

$$g_{ij} = \begin{pmatrix} -h & 0 \\ 0 & h_{ab} \end{pmatrix}, \quad (6.14)$$

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \quad (6.15)$$

where  $h = \det(h_{ab})$ . Then

$$\sqrt{-g} g^{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & h_{22} & -h_{12} \\ 0 & -h_{12} & h_{11} \end{pmatrix}, \quad (6.16)$$

so that the brane-wave equation becomes

$$-\partial_\tau^2 X^\mu + \partial_\sigma (h_{22} \partial_\sigma X^\mu - h_{12} \partial_\rho X^\mu) + \partial_\rho (h_{11} \partial_\rho X^\mu - h_{12} \partial_\sigma X^\mu) = 0 \quad (6.17)$$

( $\mu = 0, 1, \dots, 10$ ). The collapsed membrane solutions to this equation (and the gauge conditions  $g_{00} = -h$ ,  $g_{0i} = 0$ ) have the form<sup>1</sup>

$$X^\mu = X^\mu(\sigma) + p^\mu \tau, \quad p^2 = 0, \quad p \cdot \partial_\sigma X(\sigma) = 0, \quad (6.18)$$

where  $p^\mu$  is constant. For example

$$p^\mu = (p, 0, 0, \dots, 0, p), \quad X^\mu(\sigma) = (0, f_1(\sigma), f_2(\sigma), \dots, f_9(\sigma), 0) = 0, \quad (6.19)$$

i.e., a membrane collapsed to a line and moving (without changing shape) at the speed of light. The special case for which the functions  $f_1, f_2, \dots, f_9$  are all *constants* corresponds to the solution in [6] of a membrane collapsed to a point. These solutions are supersymmetric.

## 7. SEMICLASSICAL QUANTIZATION OF THE CLOSED SUPERMEMBRANE

In this section we semiclassically quantize a toroidal membrane propagating in a direct product of a  $d=4$  Minkowski spacetime and a 7-dimensional flat torus. As a classical solution we take

$$X^+ = p^+ \tau, \quad X^- = \frac{-1}{2p^+} \det h_{ab}^{\text{cl}} \tau, \quad (7.1)$$

$$X^I = V_a^I \xi^a, \quad h_{ab}^{\text{cl}} = V_a^I V_b^I, \quad (7.2)$$

$$V_1^I = (l_1 R_1, \dots, l_6 R_6, 0, 0, 0), \quad (7.3)$$

$$V_2^I = (0, \dots, 0, l_7 R_7, 0, 0), \quad (7.4)$$

$$S_{\text{cl}} = 0, \quad (7.5)$$

where  $l_1, \dots, l_7$  are winding numbers, and  $R_1, \dots, R_7$  are the radii of the 7-torus.

We define the fluctuations  $Z^I$  around this solution as

$$X^I = V_a^I \xi^a + Z^I, \quad I = 1 \dots 9. \quad (7.6)$$

Linearization of the constraint (5.7) around the classical solution (7.1)–(7.5) yields

$$\epsilon^{ab} \partial_a \dot{Z}^I V_{Ib} = 0. \quad (7.7)$$

Integration of this equation yields a result which contains a time independent

<sup>1</sup> We have been informed that a more general solution which reads:  $X^\mu = X^\mu(u(\sigma, \rho)) + P^\mu(u(\sigma, \rho))\tau$ , where  $u$  is an arbitrary function of  $\sigma$  and  $\rho$  was already discovered by I. Bars in 1975 (unpublished).

arbitrary function. This function can be set equal to zero by using the residual symmetry (5.17) completely. Thus we are led to the linearized constraint

$$\varepsilon^{ab} \partial_a Z^I V_{Ib} = 0. \quad (7.8)$$

Using (7.5)–(7.6), and (7.8), from (5.10)–(5.11) we find the linearized field equations

$$-\ddot{Z}^I + (hh^{ab})^{cl} \partial_a \partial_b Z^I = 0, \quad (7.9)$$

$$\dot{S} + \varepsilon^{ab} V_a^I \Gamma_I \partial_b S = 0. \quad (7.10)$$

The second equation can be written as a Dirac equation on the world volume for eight 2-component spinors. To do this, we decompose the 16-component  $SO(9)$  spinor  $S$  into two 8-component spinors of  $SO(7) \times SO(2)$ ,  $(S_1^A, S_2^A)$  ( $A = 1, \dots, 8$ ), and choose two of the  $SO(9)$  matrices as

$$V_a^I \Gamma_I = e_a^Q \sigma_Q \otimes I_8, \quad Q = 1, 2, 3, \quad (7.11)$$

where

$$g_{ij}^{cl} = e_i^P e_j^Q \eta_{PQ}, \quad \eta_{PQ} = \text{diag}(-1, +1, +1). \quad (7.12)$$

Suppressing  $d=3$  spinor indices, Eq. (7.10) now reads

$$e_Q^i \tau^Q \partial_i S^A = 0, \quad \tau^0 = i\sigma^2, \tau^1 = \sigma^1, \tau^2 = \sigma^3. \quad (7.13)$$

Similarly, the bosonic field equation (7.9) can also be written in a manifestly  $d=3$  covariant form

$$\partial_i \{ (\sqrt{-g} g^{ij})^{cl} \partial_j Z^I \} = 0. \quad (7.14)$$

Equations (5.5) and (5.6), which define the derivatives of  $X^-$ , will be needed later up to quadratic order in fluctuations. Recalling the constraint (6.8), it is straightforward to derive the result

$$\dot{X}^- = \frac{-1}{2p^+} \{ h^{cl} + \dot{Z}^I \dot{Z}_I + (hh^{ab})^{cl} \partial_a Z^I \partial_b Z_I + \partial_a G^a(Z) \} - i \sqrt{2} \bar{S} \dot{S}, \quad (7.15)$$

$$\partial_a X^- = \frac{-1}{p^+} \{ \dot{Z}^I V_a^I + \dot{Z}^I \partial_a Z_I \} - i \sqrt{2} \bar{S} \partial_a S, \quad (7.16)$$

where

$$G^a(Z) = (hh^{ab})^{cl} V_b^I Z_I + 2\varepsilon^{ac} \varepsilon^{bd} V_b^I Z^J V_c^J \partial_a Z_I. \quad (7.17)$$

For our classical solution we find, from (5.19), that  $\delta S = 0$  provided

$$e_3^Q \sigma_Q \alpha + 2 \sqrt{2} p^+ \beta = 0. \quad (7.18)$$

Consequently, the combination of  $\alpha$ - and  $\beta$ -transformations orthogonal to the one in (7.18) is a supersymmetry of the classical solution.

Before solving the equations of motion, it is convenient to make the redefinition

$$\chi = S_1^A + iS_2^A. \quad (7.19)$$

In this notation and suppressing the  $SO(7)$  spinor index  $A$ , the Dirac equation (7.13) becomes

$$\dot{\chi} - B \partial_\sigma \chi^* + iA \partial_\rho \chi^* = 0, \quad (7.20)$$

where

$$A \equiv \sqrt{(l_1 R_1)^2 + \cdots + (l_6 R_6)^2}, \quad B \equiv l_7 R_7. \quad (7.21)$$

The general solutions of the field equations (7.9) and (7.20) are, as in [11],

$$\begin{aligned} Z = Z_0 + \underline{p}\tau + \frac{1}{\sqrt{2}} \sum_{m^2 + n^2 \neq 0} \frac{1}{\omega_{mn}} e^{i(m\sigma + n\rho)} \\ \times [\underline{g}_{mn}^\dagger e^{i\omega_{mn}\tau} + \underline{g}_{-m-n} e^{-i\omega_{mn}\tau}], \end{aligned} \quad (7.22)$$

and

$$\begin{aligned} \chi = \sqrt{2} S_{00} + \sum_{m^2 + n^2 \neq 0} e^{i(m\sigma + n\rho)} \\ \times \left[ \frac{m - in}{\omega_{mn}} S_{mn}^\dagger e^{i\omega_{mn}\tau} + S_{-m-n} e^{-i\omega_{mn}\tau} \right], \end{aligned} \quad (7.23)$$

where

$$\omega_{mn} = \sqrt{B^2 m^2 + A^2 n^2}. \quad (7.24)$$

In order to quantize the expansion coefficients, we must take into account the fact that the fluctuations  $Z^I$  obey the constraints (7.7) and (7.8). Since these constraints are second class we apply Dirac's method of quantization. Denote the constraints (7.7) and (7.8) by  $\phi_1$  and  $\phi_2$ ,

$$\phi_1 \equiv \frac{1}{\sqrt{h^{\text{cl}}}} \varepsilon^{ab} P_a^I V_{Ib} = 0, \quad (7.25)$$

$$\phi_2 \equiv \frac{1}{\sqrt{h^{\text{cl}}}} \varepsilon^{ab} \partial_a Z^I V_{Ib} = 0, \quad (7.26)$$



where  $P' = \dot{Z}'$ . Using the Poisson bracket relations

$$\{\phi_1(\sigma, \rho), \phi_2(\sigma', \rho')\}_{\text{PB}} = \bar{\nabla}^2 \delta(\sigma - \sigma') \delta(\rho - \rho') \equiv C^{12}, \quad (7.27)$$

where  $\bar{\nabla} = (h^{ab})^{\text{cl}} \partial_a \partial_b$ , we compute the Dirac bracket of  $P'$  with  $Z'$ . After quantization this becomes the commutator

$$[P'(\sigma, \rho), Z'(\sigma', \rho')] = i(\delta^{IJ} - (h^{-1})^{\text{cl}} \varepsilon^{ab} \varepsilon^{cd} V_b^I V_d^J \partial_a \partial_c \bar{\nabla}^{-2}) \delta(\sigma - \sigma') \delta(\rho - \rho'). \quad (7.28)$$

For  $\chi$ , standard canonical quantization rules yield the anticommutator

$$\{\chi^{*A}(\sigma, \rho), \chi^B(\sigma', \rho')\} = 2i\delta^{AB} \delta(\sigma - \sigma') \delta(\rho - \rho'). \quad (7.29)$$

Substituting (7.22) and (7.23) into (7.28) and (7.29), we find the commutation relations

$$[\alpha_{mn}^I, \alpha_{m'n'}^{I\dagger}] = \omega_{mn} \{ \delta^{IJ} - \omega_{mn}^{-2} \varepsilon^{ab} \varepsilon^{cd} V_b^I V_d^J m_a m_c \} \delta_{mm'} \delta_{nn'}, \quad (7.30)$$

$$\{S_{mn}^A, S_{m'n'}^{B\dagger}\} = \delta^{AB} \delta_{mm'} \delta_{nn'}, \quad (7.31)$$

$$[P^I, Z_0^J] = -i \delta^{IJ}, \quad (7.32)$$

$$\{S_{00}^A, S_{00}^{B\dagger}\} = \delta^{AB}, \quad (7.33)$$

where  $m_a = (m, n)$ . Not all  $\alpha$ -oscillators are independent. The constraint (7.8) implies the relation

$$\varepsilon^{ab} m_a \alpha_{mn}^I V_{Ib} = 0. \quad (7.34)$$

As can be easily checked, the commutation relation (7.30) is consistent with this constraint.

To determine the mass formula, we first compute  $P^-$  from (7.15). To the order at which we are working, it is given by  $\int d\sigma d\rho \dot{X}^-$ . Thus, substituting the expansions (7.22) and (7.23) into (7.15) and integrating we obtain

$$P^- = \frac{1}{2p^+} \left\{ h^{\text{cl}} + \underline{p}^2 + \sum_{m^2 + n^2 \neq 0} [\alpha_{mn}^I \alpha_{mn}^{I\dagger} + \alpha_{mn}^{I\dagger} \alpha_{mn}^I] + \omega_{mn} (-S_{mn}^A S_{mn}^{A\dagger} + S_{mn}^{A\dagger} S_{mn}^A) \right\}. \quad (7.35)$$

Thus, the 11-dimensional mass formula is

$$(mass)^2 = (AB)^2 + 2 \sum_{m^2 + n^2 \neq 0} :(\alpha_{mn}^{I\dagger} \alpha_{mn}^I + \omega_{mn} S_{mn}^{A\dagger} S_{mn}^A):. \quad (7.36)$$

Note that the vacuum energies coming from the normal ordering of the bosonic and fermionic oscillators have cancelled: the trace of the r.h.s. of (7.30) contributes a factor  $(+9-1)\omega_{mn}$ , while the trace of the r.h.s. of (7.31) contributes  $-8\omega_{mn}$ .

Constraints analogous to the equality of the left and right Hamiltonians in the closed string theory, arise in the supermembrane case from (5.8). For the toroidal membrane, there are two noncontractable loops to consider. The result of the  $\sigma$ - and  $\rho$ -integrations in (5.8) can be summarized as

$$P^I V_a^I + N_a^{(B)} + N_a^{(F)} = 0, \quad (7.37)$$

where the number operators are

$$N_a^{(B)} = \sum_{m^2 + n^2 \neq 0} \frac{m_a}{\omega_{mn}} \alpha_{mn}^{I\dagger} \alpha_{mn}^I, \quad m_a = (m, n), \quad (7.38)$$

$$N_a^{(F)} = \sum_{m^2 + n^2 \neq 0} m_a S_{mn}^{A\dagger} S_{mn}^A. \quad (7.39)$$

These operators satisfy the commutation relations

$$[N_a^{(B)}, \alpha_{mn}^{I\dagger}] = m_a \alpha_{mn}^{I\dagger}, \quad (7.40)$$

$$[N_a^{(F)}, S_{mn}^{A\dagger}] = m_a S_{mn}^{A\dagger}, \quad m_a = (m, n). \quad (7.41)$$

Similar equations hold for  $\alpha_{mn}$  and  $S_{mn}$  with  $m_a$  replaced by  $-m_a$ . In deriving these equations, we have used the commutation relations (7.30) and (7.31), and the constraint (7.34).

Following Ref. [11], we choose the vacuum to satisfy

$$\alpha_{mn}^I |vac\rangle = 0, \quad S_{mn}^A |vac\rangle = 0, \quad m^2 + n^2 \neq 0. \quad (7.42)$$

The mass of a state obtained by acting on the vacuum with creation operators

$$\alpha_{m_1 n_1}^\dagger \cdots \alpha_{m_i n_i}^\dagger S_{p_i q_i}^\dagger \cdots S_{p_j q_j}^\dagger |vac\rangle \quad (7.43)$$

will be

$$(mass)^2 = (l_7 R_7)[(l_1 R_1)^2 + \cdots + (l_6 R_6)^2] + \omega_{m_1 n_1} + \cdots + \omega_{m_i n_i} + \omega_{p_1 q_1} + \cdots + \omega_{p_j q_j}. \quad (7.44)$$

The states (7.42) are subject to the constraint (7.37), which implies that

$$P^I V_1^I + m_1 + \cdots + m_i + p_1 + \cdots + p_j = 0, \quad (7.45)$$

$$P^I V_2^I + n_1 + \cdots + n_i + q_1 + \cdots + q_j = 0. \quad (7.46)$$

In order to find the fluctuation spectrum of the supermembrane, we also need to know the quantum numbers of the vacuum state  $|vac\rangle$ , which is not uniquely defined by (7.42). Since the fermionic oscillators  $S_{00}^A$  obey the Clifford algebra (7.33), and the membrane solution is supersymmetric, the vacuum should form a supermultiplet of the  $N=8$  super Poincaré algebra in  $d=4$ . Moreover, since the solitonic sector under consideration contains the classical mass  $(l_7 R_7)^2 [(l_1 R_1)^2 + \dots + (l_6 R_6)^2]$ , we *expect* that this multiplet is the smallest ordinary *massive*  $N=8$  supermultiplet in  $d=4$ , which consists of  $2^{15}$  bosonic and  $2^{15}$  fermionic degrees of freedom, and contains a single spin 4 state [23]. The consistency of these vacuum quantum numbers should be checked by considering the action of various Lorentz and supersymmetry generators on the vacuum, in a manner discussed by Brink for the case of the G-S superstring [24].

Note that there exists an interesting limit in which  $R_7$  goes (stepwise) to zero, while  $l_7$  goes to infinity, such that  $l_7 R_7$  is kept fixed. This corresponds to cutting the toroidal membrane along the  $\sigma$ -direction, and rolling it up along the  $\rho$ -direction to obtain a closed string. In this case all the formulae of this section go nicely to the familiar formulae for a closed type IIA G-S superstring compactified to  $d=4$  on a 6-torus.

A general feature of semiclassical quantization about classical solutions, which is illustrated by the above example, is that the existence of massless states depends on two properties: (i) the vanishing of the classical mass and (ii) supersymmetry to ensure the vanishing of the energy of the zero point fluctuations. The toroidal solution considered here satisfies (ii) but not (i). Only the collapsed membrane solutions satisfy both (i) and (ii) but there is a continuous family of such solutions. This would lead us to expect the massless spectrum of the supermembrane to be continuous. If so, this would preclude a particle interpretation. It is not clear whether this degeneracy will survive a full quantization, however.

## 8. PATH INTEGRALS AND 3-MANIFOLDS

One way to go beyond semiclassical quantization would be to write scattering amplitudes for particles coupling to the membrane in terms of a path integral over all embeddings  $X^\mu$  and all 3-metrics  $g_{ij}$  [25]. The  $X^\mu$  path integral is that of a (nonrenormalizable) sigma model (for the bosonic membrane; for the supermembrane we have additional problems of gauge fixing to face, analogous to those of the G-S superstring). The 3-metric path integral can be separated into an integration over all metrics for a 3-volume of a definite topology, followed by a sum over all topologies. Even for a 3-manifold of fixed topology the moduli space of all 3-metrics modulo 3-dimensional diffeomorphisms is infinite dimensional. Here we shall discuss only the sum over topologies.

The first point to decide is whether we should include all 3-manifolds or only orientable ones. Because of the term  $\varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho B_{\mu\nu\rho}$  in the (curved superspace) action, we must restrict ourselves to *orientable* 3-manifolds only. (This

does not mean that we restrict the membrane itself to be orientable.) The simplest amplitude that one can consider is the vacuum-persistence amplitude. In this case we would need to sum over all *compact oriented* 3-manifolds without boundaries. In this section we just review a few mathematical facts about such manifolds which may be of use in trying to carry out the calculation of the vacuum-persistence amplitude. We do not attempt to perform the calculation itself here.

All compact orientable 3-manifolds are *contact* manifolds. This is a manifold with a *contact structure*  $(J_i^j, \xi^i, \eta_j)$  satisfying [26]

$$(J^2)^j_i = -\delta^j_i + \eta_i \xi^j. \quad (8.1)$$

$(J_i^j, \xi^i, \eta_j)$  are the components of a globally defined  $(1, 1)$  tensor, vector, and 1-form, respectively; this is possible because all compact 3-manifolds have vanishing Euler number. This is analogous to the result that any 2-manifold has a complex structure  $J_i^j$  such that  $(J^2)^j_i = -\delta^j_i$ .

All compact orientable 3-manifolds can be constructed in a variety of ways, for example:

(1) Just as one can represent a surface by an appropriate polygon having pairwise associated faces, one can represent a 3-manifold  $M^3$  by a 3-dimensional full (solid) polyhedron having pairwise associated surface faces [27, 28]. There is a classical criterion which says that a complex obtained in this way is a manifold iff its Euler characteristic is zero. The simplest example is the 3-torus which can be obtained from the cube by identification of opposite sides.

(2) Start from a *knot* (closed simple curve) or *link* (union of disjoint knots). Cut out a tubular neighbourhood and glue it back with a different identification; this is called *Dehn surgery* [27, 29]. The classification of 3-manifolds is thereby related to the classification of links.

(3) Start with the 3-ball  $D^3$  and attach  $g$  copies of  $D^2 \times \{-1, 1\}$  (solid handles) with homeomorphisms mapping the  $2g$  discs  $D^2 \times \{\pm 1\}$  onto  $2g$  distinct disjoint discs on  $\partial D^3$ . The resulting space is called a *handle body* of genus  $g$ . Now glue together two handle bodies  $H_1, H_2$  of the same genus along their boundaries, by means of a homeomorphism  $h$ , to obtain a compact orientable manifold  $M^3$ , without boundary. The triple  $(H_1, H_2, h)$  is called a *Heegaard diagram* of genus  $g$  of  $M^3$ . The *genus* of a closed 3-manifold  $M^3$  is the smallest genus of all the Heegaard diagrams which yield  $M^3$  up to a homeomorphism. For example, the 3-sphere has genus zero and the 3-torus has genus three [29].

Of course, the constructions described above do not provide a *classification* of all compact 3-manifolds because they do not include a procedure for deciding whether two 3-manifolds are homeomorphic. As yet this is an unsolved problem, but important recent progress has been made by Thurston [30], who has shown that for

3-manifolds that are compact quotients of geometric spaces only eight geometries need be considered. Following an article by Scott [31] we shall sketch the ideas involved here.

Recall that the simply connected homogeneous Riemannian 2-manifolds are

$$S^2, E^2, H^2, \quad (8.2)$$

with constant positive, zero, and negative curvature, respectively. All compact 2-manifolds, up to homeomorphisms, can be obtained as the quotient of one of these "geometries" by a freely acting discrete isometry group  $\Gamma$ . We can say that every 2-manifold admits a *geometric structure* modelled on  $S^2$ ,  $E^2$ , or  $H^2$ .

Thurston considered a list of "3-geometries" analogous to that of (8.2). This list is

$$E^3, H^3, S^3, S^2 \times R, H^2 \times R, \overline{SL(2; \mathbf{R})}, Nil, Sol, \quad (8.3)$$

where  $\overline{SL(2; \mathbf{R})}$  is the universal cover of  $SL(2; \mathbf{R})$ ,  $Nil$  is the (nilpotent) Heisenberg group, and  $Sol$  is the (solvable) group of real matrices

$$\begin{pmatrix} a & 1 & b \\ 0 & a^{-1} & c \\ 0 & 0 & 1 \end{pmatrix}, \quad (8.4)$$

with  $a$  positive. Let  $X$  denote one of the 3-geometries in the list (8.3). A 3-manifold that is homeomorphic to  $X/\Gamma$ , for a freely acting discrete isometry group  $\Gamma$  of  $X$ , is said to admit a geometric structure modelled on  $X$ .

Not all compact orientable 3-manifolds without boundaries are homeomorphic to an  $X/\Gamma$ , but according to *Thurston's conjecture* [30]<sup>2</sup> they can all be obtained from the latter in the following way. Firstly, we can express any compact orientable 3-manifold without boundary as a connected sum of *irreducible* such manifolds, and a theorem of Milnor says that this decomposition is unique. According to Thurston's conjecture, each irreducible compact orientable 3-manifold without boundary *either* admits a geometric structure for a unique  $X$  and  $\Gamma$ , *or* is a finite "sum" of "octopuses" [32]. An "octopus" is a non-compact 3-manifold homeomorphic to  $X/\Gamma$  for  $X = H^3$  or  $X = H^2 \times R$ . It has no boundary but has a number of "arms" going off to infinity, which are locally  $T^2 \times R$ . Cutting off an arm produces a boundary that is homeomorphic to  $T^2$ . Two cut arms of the same or

<sup>2</sup> This conjecture includes the infamous Poincaré conjecture which essentially states that if a 3-manifold  $M^3$  is compact, with no boundary and  $\Pi_1(M^3) = 0$ , then  $M^3$  is  $S^3$ . Although the proof of the Poincaré conjecture has defied all attempts for the last 70 years, it is known to hold for 3-manifolds of Heegaard genus 1 and 2. We thank R. Lickorish for bringing these facts to our attention.

distinct “octopuses” can be glued together by identifying the two  $T^2$ -boundaries. If all the arms of a set of “octopuses” are cut and glued together in this way one obtains an orientable compact 3-manifold without boundary.

Finally we consider two well-known classes of 3-manifolds [27, 29]:

(a) *Lens Spaces*. A lens space  $L(p, q)$  ( $p, q$  integers,  $q$  prime relative to  $p$ ) can be described as follows. Consider  $S^3$  as the unit sphere in  $C^2$ . Let  $\tau: S^3 \rightarrow S^3$  be the homeomorphism

$$\tau(z_0, z_1) = (z_0 w, z_1 w^q), \quad (8.5)$$

where  $w = \exp(2\pi i/p)$  is the principal  $p$ th root of unity. Then  $\tau$  is periodic of period  $p$ , thus generating a  $Z_p$ -action on  $S^3$ . The lens space is the orbit space of this action: that is, we identify the points  $(x, y)$  of  $S^3$  if  $x = \tau^k(y)$  for some  $k$ . The following homeomorphisms exist:

$$\begin{aligned} L(1, q) &\cong S^3, \\ L(0, 1) &\cong S^2 \times S^1, \\ L(2, 1) &\cong RP^3. \end{aligned} \quad (8.6)$$

The lens spaces have been completely classified. They are in fact precisely the 3-manifolds of Heegaard genus 1. Two lens spaces  $L(p, q)$ ,  $L(p', q')$  are homeomorphic if

$$\pm q' \equiv q^{\pm 1} \pmod{p}, \quad (8.7)$$

and are of the same homotopy type if

$$\pm qq' \equiv m^2 \pmod{p}, \quad (8.8)$$

for some integer  $m$ . The fundamental group of  $L(p, q)$  is  $Z_p$ .

(b) *Seifert Manifolds*. These were defined and classified by Seifert in 1933. Two equivalent definitions are *either* as 3-manifolds that can be foliated by circles, *or* as  $U(1)$  bundles over a 2-dimensional orbifold [31]. The importance of Seifert spaces is that all of the compact 3-manifolds that admit a geometric structure modelled on any of the eight Thurston geometries except  $H^3$  or  $Sol$  are Seifert spaces. Moreover the geometry is determined by two topological invariants, the Euler number of the base orbifold and the Euler number of the  $U(1)$  bundle. Other useful references on 3-manifolds are [33] and further references can be found in [31].

## 6. CONCLUSIONS

String theory is potentially a solution to the vexing problem of quantum gravity. Within the context of purely bosonic theories there would not appear to be any advantage to proceeding on the higher-dimensional extended objects. "What is the problem for which membrane theory provides a solution?" is an often asked question. For the bosonic membrane we have no answer to this question (except to observe that "cosmic" membranes can arise as classical extended object solutions of certain field theories, but here we are concerned with a fundamental membrane theory.) The supermembrane, however, is potentially a solution to the "enigma" of 11-dimensional supergravity. There is a finite "constellation" of super  $p$ -brane actions all related to the  $d=3, 4, 6, 10$  superstring theories, and recent results indicate that only the  $d=10$  superstring and the  $d=11$  supermembrane are candidates for a fundamental theory. Much work needs to be done before we shall know whether the  $d=11$  supermembrane can be consistently quantized and, if so, whether it provides a quantum consistent extension of  $d=11$  supergravity. Nevertheless, the results obtained so far are encouraging and quite interesting in their own right.

The supermembrane action, like the G-S superstring action, has manifest spacetime supersymmetry. It is therefore remarkable that, considered as a 3-dimensional field theory, it should have an equal number of bosons and fermions. These "fermions" are spacetime spinors, a priori, but become world volume spinors on fixing the reparametrization invariance by a "physical" gauge choice. In addition, for a flat spacetime background, the spacetime supersymmetry and  $\kappa$ -invariance combine, in this gauge, to yield eight world volume supersymmetries. Thus, the  $d=11$  supermembrane in flat superspace can be thought of (at least on a coordinate chart of the world volume) as an  $N=8$  ( $8+8$ )-component 3-dimensional supersymmetric field theory (which is not of sigma model form, however). It is important to realize that this is a rigid supersymmetry. At present there is no acceptable "Neveu-Schwarz-Raymond" (NSR) formulation of the supermembrane incorporating local world volume supersymmetry, i.e., 3-dimensional supergravity. An attempt to construct such an action was made in [34], but this attempt is not successful. The problem is that the supersymmetrization of the 3-dimensional cosmological constant requires an Einstein-Hilbert term in the action, rendering it inequivalent to the bosonic membrane action in the limit that all fermions vanish. The problem does not arise for the superstring because the cosmological term is absent.

It may be, of course, that other supermembrane actions, containing higher-derivative terms, exist for which there is an NSR formulation. It is not easy to find  $\kappa$ -invariant extension of the supermembrane action, however. As yet there is no "tensor calculus" that would enable us to write down manifestly  $\kappa$ -invariant actions. An attempt to introduce higher-derivative terms into the G-S action has been made recently [35] but no definitive results have been obtained yet.

Very recently significant progress has been made on the covariant first quan-

tization of the G-S superstring [36] and these methods may be applicable to the supermembrane. The Hamiltonian formulation of the supermembrane action has also been worked out [37] and a natural proposal for a quantum membrane equation is the analogue of the Wheeler-DeWitt equation for gravity, i.e.,  $H\Psi=0$ , where  $H$  is the Hamiltonian constraint operator and  $\Psi$  is a wave-functional of 2-surfaces. This equation is only linear in  $\Psi$ , of course but, given the non-linearity of  $H$ , perhaps this is sufficient. In this connection we remark that Biran *et al.* [38] have argued that the way in which membranes are an extension of strings is similar to the way in which the Yang-Mills equations are an extension of Maxwell's equations.

In the Polyakov path integral approach to the quantization of extended objects we have to face the problem of nonrenormalizability of 3-dimensional sigma models. As for strings, the nonrenormalizable interactions of the sigma model may have a physical interpretation as vertex operators of particles in the spectrum, so this problem seems less severe than those raised by the sum over all 3-surfaces. Here we have to worry about curvature terms appearing from quantum corrections, in which case the auxiliary 3-metric might cease to be auxiliary. There is also the problem of how to order the perturbation series.

So far we have discussed only whether the 11-dimensional supermembrane might be consistently quantized. Whether it could ever be the basis of a phenomenologically acceptable particle physics model is another question. One obvious problem that the supermembrane would appear to share with 11-dimensional supergravity is the lack of chirality. Recently, however, it has been shown how the non-chiral type IIA superstring can yield a chiral low energy effective theory by means of an intrinsically "stringy" mechanism [39]. Perhaps there are similar intrinsically "membrany" mechanisms waiting to be discovered.

In this paper we have emphasized those aspects of membranes and supermembranes that initially motivated us, and that are of particular current interest (see [40] for a recent review). There are other aspects, however. One can imagine that spacetime is itself the world volume of a 3-brane, for example. This has been discussed recently in [4, 41]. An older application is to the theory of relativistic bubbles and bag models for hadrons [42, 43]. Membranes are of course relevant to the theory of domain walls; fields coupled to domain walls that have modes trapped on the wall can give rise to interesting effects [44], for example. Other aspects of the quantum mechanics of membranes, e.g., BRST quantization, can be found in Ref. [45-49], and other aspects of classical membranes in [50-52].

## APPENDIX: NOTATION AND CONVENTIONS

### (i) General Conventions

Signature of  $g^{ij} = (-, +, +, +)$ ,  $g = \det g_{ij}$ .

Signature of  $\eta_{\mu\nu} = (-, +, +, +, +, +, +, +, +, +, +)$ .



$\varepsilon_{012} = -\varepsilon^{012} = 1$ ;  $1/\sqrt{-g} \varepsilon^{ijk}$ , and  $\sqrt{-g} \varepsilon_{ijk}$  are tensors.

$\varepsilon^{ijk} \varepsilon^{mnp} = -(\sqrt{-g})^2 (g^{im} g^{jn} g^{kp} + 5 \text{ more terms})$ .

$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ ,  $\Gamma^{\mu\dagger} = \Gamma^0 \Gamma^\mu \Gamma^0$ .

$\theta^\alpha = C^{\alpha\beta} \theta_\beta$ ,  $\theta_\alpha = \theta^\beta C_{\beta\alpha}$ ,  $C_{\alpha\beta} = -C_{\beta\alpha}$ ,  $C^{\alpha\beta} C_{\beta\gamma} = \delta^\alpha_\gamma$ .

$\bar{\theta} = \theta^\dagger \Gamma_0$ ,  $(\bar{\chi} \Gamma^{v_1 \dots v_n} \lambda)^\dagger = -(\bar{\chi} \Gamma^{v_1 \dots v_n} \lambda)$  for anticommuting  $\chi, \lambda$ .

$\bar{\chi} \lambda = \chi_\alpha \lambda^\alpha$ ,  $\bar{\chi} \Gamma_\mu \lambda = \chi_\alpha (\Gamma_\mu)^\alpha_\beta \lambda^\beta$ , etc.

(Anti)symmetrization is with unit strength, e.g.,  $\Gamma^{\mu\nu} = \frac{1}{2}(\Gamma^\mu \Gamma^\nu - \Gamma^\nu \Gamma^\mu)$ .

$(\Gamma^\mu)_{\alpha\beta}$ ,  $(\Gamma^{\mu\nu})_{\alpha\beta}$  are symmetric,  $(\Gamma^{\mu\nu\rho})_{\alpha\beta}$  is antisymmetric.

### (ii) Superspace Conventions

Superspace coordinates are  $Z^M = (X^\mu, \theta^{\dot{a}})$ , supervielbein is  $E^A_M (A = a, \alpha)$ .

$E^A_M E^N_A = \delta^N_M$ ,  $E^A_M E^B_M = \delta^B_A$ .

$V^A = V^M E^A_M$ ,  $V_A = E^M_A V_M$ .

$V^M = V^A E^M_A$ ,  $V_M = E^A_M V_A$ .

$E^A = dZ^M E^A_M$  ( $E^a E^b = -E^b E^a$  but  $E^\alpha E^\beta = E^\beta E^\alpha$ ).

$F = (1/p!) E^{A_1} \dots E^{A_p} F_{A_1 \dots A_p} = (1/p!) dZ^{M_1} \dots dZ^{M_p} F_{M_1 \dots M_p}$ .

$dF = (1/p!) dZ^{M_1} \dots dZ^{M_p} dZ^N (\partial/\partial Z^N) F_{M_1 \dots M_p}$ .

$d(FG) = FdG + (-1)^q dFG$  for  $p$ -form  $F$  and  $q$ -form  $G$ .

$H = dB$ ;  $H_{MNPQ} = \partial_M B_{NPQ} + 3 \text{ more terms}$ .

$T^A = dE^A + E^B \Omega^A_B$ ;  $T^A_{MN} = \partial_M E^A_N + (-1)^{m(b+n)} E^B_N \Omega^A_{MB} + 1 \text{ more term}$ , where  $m=0$  (1) if  $M$  = vector (spinor) index.

### (iii) Lightcone Conventions

$V^\pm = (1/\sqrt{2})(\pm V^0 + V^{10})$ .

$V^\mu W_\mu = V^I W^I + V^+ W^- - V^- W^+$ ,  $I = 1 \dots 9$ .

$\varepsilon^{0ab} = -\varepsilon^{ab}$ ,  $\varepsilon^{ab} \varepsilon^{cd} = h(h^{ac} h^{bd} - h^{bc} h^{ad})$ ,  $a = 1, 2$ .

$\Gamma^\mu = (\Gamma^I, \Gamma^+, \Gamma^-)$ , with  $\Gamma^I = \gamma^I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

$\Gamma^+ = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^{10}) = I_{16} \otimes \begin{pmatrix} 0 & 0 \\ \sqrt{2}i & 0 \end{pmatrix}$ .

$\Gamma^- = \frac{1}{\sqrt{2}}(-\Gamma^0 + \Gamma^{10}) = I_{16} \otimes \begin{pmatrix} 0 & \sqrt{2}i \\ 0 & 0 \end{pmatrix}$ .

$\{\gamma^I, \gamma^{J'}\} = 2\delta^{IJ}$ ,  $\{\Gamma^+, \Gamma^-\} = 2$ ,  $(\Gamma^+)^2 = (\Gamma^-)^2 = 0$ .

$\theta = (i\theta_1, \theta_2)$ ,  $\bar{\theta} = (-i\bar{\theta}_2, -\bar{\theta}_1)$ .

$\bar{\theta}_1 \chi_1 = -\bar{\chi}_1 \theta_1$ ,  $\bar{\theta}_1 \gamma^I \chi_1 = -\bar{\chi}_1 \gamma^I \theta_1$ ,  $\bar{\theta}_1 \chi^{IJ} \chi_1 = +\bar{\chi}_1 \gamma^{IJ} \theta_1$ , etc.

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## NOTE ADDED IN PROOF:

## RECENT DEVELOPMENTS IN MEMBRANE AND SUPERMEMBRANE THEORY

Since the completion of this paper many others have been written on membranes and supermembranes, and some older ones have been brought to our attention. For the convenience of the reader we present here a very brief overview of some of these works.

1. *Classical Bosonic Membranes*

Dolan and Tchrakian have proposed [53] an alternative bosonic  $p$ -brane action with an independent world volume metric but without the cosmological term. (See also [86].) For the membrane ( $p = 2$ ) their action reads

$$I = -\frac{1}{2} \int d^3\xi \sqrt{-g} \{ g^{ij} M_{ij} + c g^{i[j} g^{k]l} M_{ij} M_{kl} \}, \quad (10.1)$$

where  $c$  is a constant and  $M_{ij} \equiv \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$  is the metric induced by the spacetime metric  $g_{\mu\nu}$ . The equation for the independent world volume metric  $g_{ij}$  is a matrix equation for which *one* solution is  $g_{ij} \propto M_{ij}$ . Substituting this solution into the action one reobtains the Dirac membrane action. The action (10.1) might be thought to provide a starting point for an "NSR formulation" of the supermembrane, i.e., a "spinning membrane." This possibility was investigated in [54] but a recent study [55] has shown that such models cannot be constructed within the framework of the 3-dimensional supergravity tensor calculus.

2. *Classical Supermembrane*

The super  $p$ -brane action requires the existence of a superspace  $(p+2)$ -form  $H$  that is closed,  $dH = 0$ . *Locally*, therefore,  $H = dB$ . In *flat* superspace one expects that  $B$  exists globally, as is certainly so for the 11-dimensional supermembrane (with  $B$  given in (4.2)–(4.4)). A proof for arbitrary (allowed)  $(p, d)$  has been given by Evans [56], who has also derived a general formula for the  $(p+1)$ -form  $B$ . In our conventions it reads

$$C_p \left\{ \sum_{r=0}^p \binom{p+1}{r+1} \Pi^{a_p} \dots \Pi^{a_{r+1}} (id\bar{\theta} \Gamma^{a_r} \theta) \dots (id\bar{\theta} \Gamma^{a_1} \theta) \right\} (id\bar{\theta} \Gamma_{a_1 \dots a_p} \theta), \quad (10.2)$$

where  $C_p$  is a  $p$ -dependent, but  $d$ -independent, constant.

The gauge fixing procedure for the “physical” gauge described in Section 4 has now been carried to completion for the  $d=4$  supermembrane [57]. The gauge fixed 3-dimensional action is a functional of the single scalar field  $X(\xi)$  and the single 2-component real spinor field  $\psi(\xi)$ , and reads

$$I = - \int d^3\xi \left\{ \det^{1/2} \left[ \delta_j^i + \partial^i X \partial_j X - i \bar{\psi} (\gamma^i \partial_j + \gamma_j \partial^i) \psi - \frac{3}{2} (\bar{\psi} \psi) (\partial^i \bar{\psi} \partial_j \psi) \right] \right. \\ \left. - i \epsilon^{ijk} \partial_i X \bar{\psi} \gamma_j \partial_k \psi \right\}. \quad (10.3)$$

For  $X=0$  this action reduces to the (3-dimensional) Volkov–Akulov action for a nonlinearly realized supersymmetry with  $\psi$  the Goldstone fermion. In addition to the nonlinearly realized supersymmetry the action (10.5) has a *linearly realized* supersymmetry with  $X$  and  $\psi$  the physical components of a scalar supermultiplet. Thus “supersymmetry on the brane” has been explicitly verified for this model.

The geometry of supermembranes has been further investigated by Curtright [58] from an “extrinsic” point of view, generalising the results of [35]. Güven has found new classical solutions of the  $d=11$  supermembrane equations about black hole backgrounds [59].

### 3. Dirac's Charged Membrane

An older work developing Dirac's ideas in the context of superconducting extended objects is that of Skagerstam and Stern [60]. Recent work on the Dirac charged membrane, or generalisations of it, has been done by Tucker [61] and Onder and Tucker [62]. We also mention here that the uncharged spherical membrane solution of [18] has been generalised to arbitrary  $p$  [63].

### 4. Supermembranes and Supersingletons

The singleton actions on  $S^p \times S^1$  (the boundary of  $(AdS)_{p+2}$ ) have been constructed [64] (the result for  $p=2$  has been independently obtained in [65]) and a connection between all supersymmetric models of this type and all super  $p$ -branes (in an  $(AdS)_{p+2} \times S^{d-p-2}$  background) has been conjectured [64, 65]. (For  $p=2$ , see also [22].) In the case of the  $N=8$  supersingleton theory formulated on  $S^2 \times S^1$ , the spectrum has been found and the  $OSp(8|4)$  super- $(AdS)_4$  algebra has been shown to be anomaly-free in the quantum theory [66, 67].

### 5. First Quantized Membranes

The Casimir energy for a spherical membrane (in  $d=4$ ) as a function of radius,  $r$ , has been calculated by Sawhill [68] with the interesting result that the Casimir energy provides a repulsive force that stabilizes the membrane at a nonzero radius  $r_0$ , but the net energy of such a quantum stabilized membrane is negative

suggesting that the membrane ground state is tachyonic. Since the Casimir energy is likely to vanish for the supermembrane it seems unlikely that these results will carry over to that case.

Luckock and Moss have computed counterterms for the bosonic  $p$ -branes, viewed as  $(p+1)$ -dimensional gravity-coupled field theories [69]. They conclude that bosonic  $p$ -branes are nonrenormalizable for  $p \geq 2$ . It would be interesting to know the analogous result for the super  $p$ -branes. Kubo has argued that the spacetime dimension  $d=27$  is "critical" for the bosonic membrane [70]. His argument is based on the claim that in  $d=27$  the narrow membrane limit for a membrane compactified on a torus has a spectrum that includes massless particles. Further evidence for  $d=27$  as the critical dimension of the bosonic membrane has been found by Marquard and Scholl [71] by demanding the cancellation of certain divergent central extension terms in the algebra of the quantum constraints. Hoppe has shown that the spherical membrane can be thought of as an  $SU(\infty)$  gauge theory [16].

Recently, in an interesting paper, Hoppe, De Wit, and Nicolai [72] investigated the existence of massless states in the spectrum of an 11-dimensional supermembrane in a Minkowski background. They tried to explicitly construct the ground state wavefunction corresponding to a *nonperturbative* massless state. So far they have not been able to find any square-integrable zero energy wavefunction. Their results, however, are inconclusive.

## 6. Second Quantized Membranes

Fujikawa has recently attempted [73] to construct a lightcone membrane field theory à la Mandelstam. Ho and Hosotani have proposed [74] a type of membrane field theory equation of motion by following Dirac's derivation of the Dirac equation for particles. They have shown that this equation yields massless particles for a  $d=4$  toroidal membrane. Gamboa and Ruiz-Altaba have also proposed a "functional diffusion" equation for quantum membranes [75].

## 7. Infinite-Dimensional Algebras from (Super) Membranes

Among the infinitesimal diffeomorphisms of a membrane, the Jacobian preserving ones, i.e., those for which the parameter  $\eta^i$  satisfies

$$\eta^0 = 0, \quad \dot{\eta}^a = 0, \quad \partial_a \eta^a = 0, \quad (10.4)$$

form a closed subalgebra. These are area-preserving transformations, the area being the metric-independent symplectic form  $d\sigma^1 d\sigma^2$ . For a toroidal membrane the generators of such diffeomorphisms consist of  $L_{\mathbf{m}}$  and  $\mathbf{P}$  where the  $\mathbf{P} = (P_1, P_2)$  are the generators of constant shifts around the homology cycles of the torus, and  $\mathbf{m}$  is

an integer 2-vector. Floratos and Illiopoulos have shown that the algebra allows a central extension [76]. This centrally extended algebra is

$$\begin{aligned}[L_m, L_n] &= \mathbf{m} \times \mathbf{n} L_m + \mathbf{a} \cdot \mathbf{m} \delta_{\mathbf{m} + \mathbf{n}, 0} \\ [\mathbf{P}, L_m] &= \mathbf{m} L_m, \quad [P_1, P_2] = 0,\end{aligned}\tag{10.5}$$

where  $\mathbf{m}$  is a constant 2-vector. They have also shown for  $\mathbf{a} = 0$  that this algebra contains the Virasoro algebra (with zero central extension) as a subalgebra. More recently it has been shown that for any fixed  $\mathbf{a} \neq 0$ , the algebra (10.5) contains the Virasoro algebra with an arbitrary central extension (including zero) [77]. For  $\mathbf{a} = 0$ , a field theoretic realization of the algebra (10.5) has been discussed in [37, 72]. The central extensions of the area-preserving membrane algebras for a membrane of arbitrary topology were recently found in [78].

#### 8. Miscellaneous

In  $d = 4$  the coupling

$$\int d^3\xi \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho B_{\mu\nu\rho}(X)\tag{10.6}$$

of a third rank antisymmetric tensor  $B$  to a membrane implies that the membrane is a domain wall separating regions for which  $\varepsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho\sigma}$  takes on different values. Brown and Teitelboim have suggested [79] that the spontaneous quantum creation of membranes could lead to a mechanism whereby the cosmological constant can decay to zero. (For a recent discussion of the relation between Wess–Zumino terms of the type (10.6) and Hopf maps, see [87].) On a more speculative note, Gervais has recently suggested that  $p$ -adic analysis might be important for membrane theories [80], and it has even been speculated [35] that supermembranes might be relevant to biology.

For completeness we record here that several reviews/conference reports on supermembranes have recently appeared [81–85].

#### REFERENCES

1. M. B. GREEN, J. H. SCHWARZ, AND E. WITTEN, "Superstring Theory," Vol. II, p. 314, Cambridge Univ. Press, London/New York, 1987.
2. E. BERGSHOEFF, E. SEZGIN, AND P. K. TOWNSEND, *Phys. Lett. B* **189** (1987), 75.
3. M. B. GREEN AND J. H. SCHWARZ, *Phys. Lett. B* **136** (1984), 367; *Nucl. Phys. B* **243** (1984), 285.
4. J. HUGHES, J. LIU, AND J. POLCHINSKY, *Phys. Lett. B* **180** (1986), 370.
5. M. J. DUFF, P. S. HOWE, T. INAMI, AND K. S. STELLE, *Phys. Lett. B* **191** (1987), 70.
6. I. BARS, C. N. POPE, AND E. SEZGIN, *Phys. Lett. B* **198** (1987), 455.
7. A. ACHÚCARRO, J. M. EVANS, P. K. TOWNSEND, AND D. L. WILTSHIRE, *Phys. Lett. B* **198** (1987), 441.
8. G. SIERRA, *Class. & Quantum Grav.* **4** (1987); J. M. EVANS, *Nucl. Phys. B* **298** (1988), 92.

9. J. HUGHES AND J. POLCHINKI, *Nucl. Phys. B* **278** (1986), 147; P. K. TOWNSEND, *Phys. Lett. B* **202** (1988), 53.
10. I. BARS, "First Massive Level and Anomalies in the Supermembrane," preprint, USC-87/HEP06, to appear in *Nucl. Phys. B*; I. BARS AND C. N. POPE, "Anomalies in Super  $p$ -branes," preprint, USC-88/HEP02.
11. M. J. DUFF, T. INAMI, C. N. POPE, E. SEZGIN, AND K. S. STELLE, *Nucl. Phys. B* **297** (1988), 515.
12. L. MEZINCESCU, R. I. NEPOMECHIE, AND P. VAN NIEUWENHUIZEN, "Do Supermembranes Contain Massless Particles?" preprint UMTG-139, ITP-SB 87 43, 1987.
13. P. A. M. DIRAC, *Proc. R. Soc. London Ser. A* **268** (1962), 57.
14. E. CREMMER AND S. FERRARA, *Phys. Lett. B* **91** (1980), 61.
15. L. BRINK AND P. S. HOWE, *Phys. Lett. B* **91** (1980), 384.
16. J. HOPPE, "Quantum Theory of a Relativistic Surface," Aachen preprint, PITHA 86/24 and Ph.D. thesis, MIT, 1982.
17. G. K. SAVVIDY, "The Lightcone Gauge in the Theory of Relativistic Surfaces," Yerevan Physics Institute preprint 972 (22) 87, 1987.
18. P. A. COLLINS AND R. W. TUCKER, *Nucl. Phys. B* **112** (1976), 150.
19. K. KIKKAWA AND M. YAMASAKI, *Prog. Theor. Phys.* **76** (1988), 379.
20. M. E. LAZIEV AND G. K. SAVVIDY, *Phys. Lett. B* **198** (1988), 451.
21. J. HOPPE AND H. NICOLAI, *Phys. Lett. B* **196** (1987), 451.
22. E. BERGSHOEFF, M. J. DUFF, C. N. POPE, AND E. SEZGIN, *Phys. Lett. B* **199** (1987), 69; E. BERGSHOEFF, M. J. DUFF, C. N. POPE AND E. SEZGIN, "New Supermembrane Vacua and the Issue of Singletons," in preparation.
23. P. S. HOWE, K. S. STELLE, AND P. K. TOWNSEND, *Nucl. Phys. B* **192** (1981), 332.
24. L. BRINK, in "NATO Conference on Supersymmetry," (K. Dietz, R. Flume, G. van Gehlen, and V. Rittenberg, Eds.), Plenum, New York, 1985.
25. A. SUGAMOTO, *Nucl. Phys. B* **215** (1983), 381.
26. K. YANO AND M. KOU, "Structures on Manifolds," World Scientific, Singapore, 1984.
27. H. SEIFERT AND W. THREFFALL, "A Textbook of Topology," Academic Press, San Diego, CA, 1980.
28. W. P. THURSTON, "The Geometry and Topology of 3-Manifolds," Princeton Univ. Press, Princeton NJ, in press.
29. D. ROLFSEN, "Knots and Links," Publish or Perish, Houston, TX, 1976.
30. W. P. THURSTON, *Bull. Amer. Math. Soc.* **6** (1982), 357.
31. P. SCOTT, *Bull. London Math. Soc.* **15** (1983), 401.
32. S. LANG, "The Beauty of Doing Mathematics: Three Public Dialogues," (Springer-Verlag, Berlin, 1985).
33. G. HEMPEL, "3-Manifolds," Princeton Univ. Press, Princeton, NJ/University of Tokyo Press, Tokyo, 1976; P. ORLIK, "Seifert Manifolds," Springer-Verlag, Berlin, 1972; K. JOHANNSON, "Homotopy Equivalence of 3-Manifolds with Boundaries," Lecture Notes in Mathematics, Vol. 761, Springer-Verlag, Berlin, 1979.
34. P. S. HOWE AND R. W. TUCKER, *J. Phys. A* **10** (1987), L155; *J. Math. Phys.* **19** (1978), 869, 981.
35. T. L. CURTRIGHT AND P. VAN NIEUWENHUIZEN, *Nucl. Phys. B* **294** (1987), 125.
36. E. NISSIMOV, S. PACHEVA, AND S. SOLOMON, *Nucl. Phys. B* **297** (1988), 349.
37. E. BERGSHOEFF, E. SEZGIN, AND Y. TANII, *Nucl. Phys. B* **298** (1988), 187.
38. B. BIRAN, E. G. F. FLORATOS, AND G. K. SAVVIDY, *Phys. Lett. B* **198** (1988), 329.
39. L. J. DIXON, V. S. KAPLUNOVSKY, AND C. VAFA, *Nucl. Phys. B* **294** (1987), 43.
40. M. J. DUFF, *Class. & Quantum Grav.* **5** (1988), 189.
41. G. W. GIBBONS AND D. L. WILTSHIRE, *Nucl. Phys. B* **287** (1987), 717.
42. A. AURILIA, D. CHRISTODOULOU, AND F. LEGOVINI, *Phys. Lett. B* **73** (1978), 429; A. AURILIA AND D. CHRISTODOULOU, *Phys. Lett. B* **78** (1978), 589; A. AURILIA AND F. LEGOVINI, *Phys. Lett. B* **67**, 299; A. AURILIA AND D. CHRISTODOULOU, *J. Math. Phys.* **20** (1979), 1446.
43. I. YA. AREF'eva, *Phys. Lett. B* **95** (1980), 269.
44. R. MACKENZIE, "Topological Structures on Domain Walls," DAMPT preprint, 87-2, 1987.

45. M. HENNEAUX, *Phys. Lett. B* **120** (1985), 179; *Phys. Rep.* **126** (1985), 63.
46. T. E. FRADKIN, *Sov. Phys. J.* **12** (1983), 40.
47. E. S. FRADKIN AND A. A. TSEYTLIN, *JETP Lett.* **41** (1985), 206.
48. C. M. SOTKOV AND R. P. ZAIKOV, *C. R. Acad. Bulg. Sci.* **32** (1979), 1187.
49. K. FUJIKAWA AND J. KUBO, *Phys. Lett. B* **199** (1987), 75.
50. T. INAMOTO, "The BRST Hamiltonian Method and the Membrane Model," Scientific University of Tokyo preprint, 1987.
51. H. KAMO AND T. SOHKAWA, *Prog. Theor. Phys.* **57** (1977), 1749.
52. C. TEITELBOIM, *Phys. Lett. B* **167** (1986), 63, 69.
53. B. P. DOLAN AND D. H. TCHRAKIAN, *Phys. Lett. B* **198** (1987), 447.
54. C. CASTRO, "A Supersymmetric Lagrangian for the Spinning Membrane," University of Texas at Austin preprint, 1988.
55. E. BERGSHOEFF, E. SEZGIN, AND P. K. TOWNSEND, *Phys. Lett. B*, in press.
56. J. EVANS, "Super  $p$ -brane Wess-Zumino Terms," Cambridge preprint, 1988.
57. A. ACHÚCARRO, J. GAUNTLETT, K. ITOH, AND P. K. TOWNSEND, in preparation.
58. T. CURTIGHT, "Extrinsic Geometry of Superimmersions," CERN preprint TH.4924/87.
59. R. GUVEN, "Supermembranes on Blackholes," TUBITAK preprint, 1988.
60. B. S. SKAGERSTAM AND A. STERN, *Phys. Rev. D* **25** (1982), 25.
61. R. TUCKER, "Semiclassical Treatment of the Charged Membrane," Lancaster preprint.
62. M. ONDER AND R. TUCKER, *Phys. Lett. B* **202** (1988), 501.
63. B. SAWHILL, *Phys. Lett. B* **203** (1988), 69.
64. H. NICOLAI, E. SEZGIN, AND Y. TANII, "Conformally Invariant Supersymmetric Field Theories on  $S^p \times S^1$  and Super  $p$ -branes," Trieste preprint ICTP/88/7, 1988.
65. M. BLENOWE AND M. J. DUFF, *Phys. Lett. B* **203** (1988), 229.
66. E. BERGSHOEFF, ABDUS SALAM, E. SEZGIN, AND Y. TANII, *Phys. Lett. B* **205** (1988), 237.
67. E. BERGSHOEFF, ABDUS SALAM, E. SEZGIN, AND Y. TANII, " $N=8$  Supersingleton Quantum Field Theory," Trieste preprint, IC/88/5, 1988.
68. B. SAWHILL, "Nontrivial Ground State of the Closed Bosonic Membrane," Karlsruhe preprint, 1987.
69. H. LUCKOCK AND I. MOSS, "Quantum Geometry of Bosonic  $p$ -branes," Newcastle preprint NCL-87 TP12, 1988.
70. J. KUBO, *Phys. Lett. B* **202** (1988), 315.
71. U. MARQUARD AND M. SCHOLL, "Conditions on the Embedding Space of  $p$ -branes from Their Constraint Algebra," Karlsruhe preprint KA-THEP-1987-13.
72. J. HOPPE, B. DE WIT, AND H. NICOLAI, "On the Quantum Mechanics of Supermembranes," Karlsruhe preprint, KA-THEP 6/88, 1988; B. DE WIT, "Quantum Mechanics of a Supermembrane," Utrecht preprint, THU-88-15, 1988.
73. K. FUJIKAWA, "Interacting Membrane Picture in the Light-Cone Gauge," Hiroshima preprint RRK-88-3, 1988.
74. C. L. HO AND Y. HOSOTANI, *Phys. Rev. Lett.* **60** (1988), 885.
75. J. GAMBOA AND M. RUIZ-ALTABA, *Phys. Lett. B* **205** (1988), 245.
76. E. G. FLORATOS AND J. ILLIOPOULOS, *Phys. Lett. B* **201** (1988), 237.
77. I. ANTONIADIS, P. DITSAS, E. FLORATOS, AND J. ILLIOPOULOS, "New Realizations of Virasoro Algebras as Membrane Symmetries," preprint CRETE-TH-88-2, 1988.
78. I. BARS, C. N. POPE, AND E. SEZGIN, "Central Extensions of Area-Preserving Membrane Algebras," Trieste preprint, IC/88/77.
79. J. D. BROWN AND C. TEITELBOIM, *Phys. Lett. B* **195** (1987), 177; *Nucl. Phys. B* **297** (1988), 787.
80. J. L. GERVAIS, *Phys. Lett. B* **201** (1988), 306.
81. K. S. STELLE AND P. K. TOWNSEND, in "Proceedings of CAP 1987 Summer School," Edmington.
82. K. FUJIKAWA, "Path Integral, BRS Symmetry and String and Membrane Theories," Hiroshima preprint RRK-87-20, 1987.
83. C. POPE, in "Proceedings of 1987 Trieste Summer School in High Energy Physics and Cosmology" (G. Furlan et al., Eds.), World Scientific, 1988.

- 84. L. MEZINCESCU, in "Proceedings of Boulder Workshop," 1987.
- 85. P. VAN NIEUWENHUIZEN, "Superstrings, Supermembranes and Supergravity," Stony Brook preprint ITP-88-88-1, 1988.
- 86. T. INAMI AND S. YAHIKOZAWA, unpublished; M. S. ALVES AND J. BARCELOS-NETO, "Conformally Invariant Action for  $n$ -branes without Cosmological Constant," preprint IF/UFRJ/88/3, 1988.
- 87. Y. S. WU AND A. ZEE, "Membranes, Higher Hopf Maps and Phase Interactions," preprint NSF-ITP-88-34, 1988.